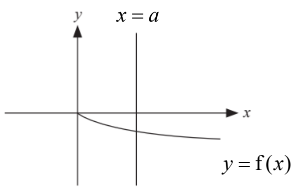
Follow up solutions C1 (1-10)

Follow up 1

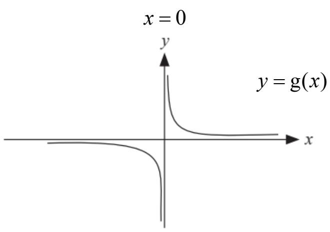
Solution

(a)



Every vertical line $x = a$, $a \geq 0$, cuts the graph of f at exactly one point. Hence, f is a function.

(b)



The vertical line $x = 0$ does not cut the graph of g at any point. Hence, g is not a function.

Follow up 2

Solution

(a) $\ln(x-1)$ is defined for $x-1>0$

$$x>1$$

$\therefore$ Maximal Domain, $\mathrm{D_f} = (1, \infty)$ or $\mathrm{D_f} = (1, \infty)$

(b) $\sqrt{3-x}$ is defined for $3-x \geq 0$

$3-x \geq 0$

$x-3 \leq 0$

$x \leq 3$

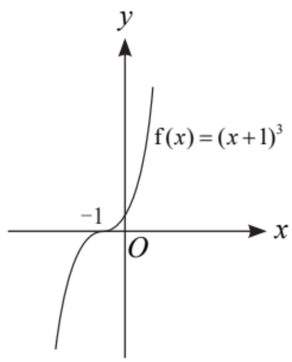
$\therefore$ Maximal Domain, $\mathrm{D_g} = (-\infty, 3]$.

(c) $\dfrac{1}{x}$ is defined if $x \neq 0$.

$\therefore$ Maximal Domain, $\mathrm{D_h} = \mathbb{R} \setminus \{0\}$.

Follow up 3

Solution



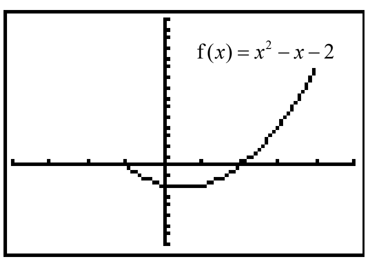
$\therefore$ the new function g(x) and its domain as follows:

$g : x \mapsto (x+1)^3, x \in \mathbb{R}$ and $x > -1$

Follow up 4

Solution

Graphical Method



From GC, the range of $f(x)$ is $[-2.25, 10]$.

Analytical Method

Let $y = f(x) = x^2 - x - 2$ for $-1 \le x \le 4$

When $x = -1$, $y = 0$

When $x = 4$, $y = 10$

Completing the square,

$$y = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$$

As $\left(x - \frac{1}{2}\right)^2 \ge 0$ for all real values of x,

$$y = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4} \ge -\frac{9}{4}$$

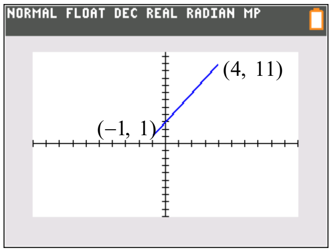
$\therefore$ the minimum value of f is $-\frac{9}{4}$.

Hence, the range of f is $R_f = [-2.25, 10]$

Follow up 5

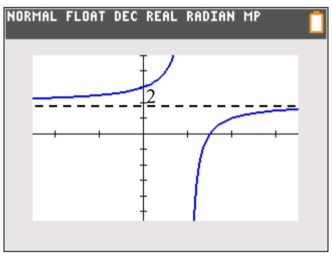
Solution

(a)



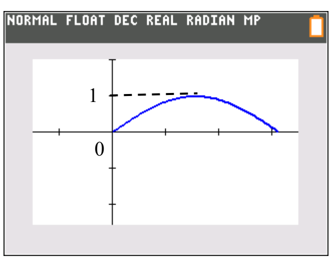
From the graph, the range of f is $R_f = (1, 11)$.

(b)



From the graph, the range of g is $R_g = \mathbb{R} \setminus \{-2\}$.

(c)

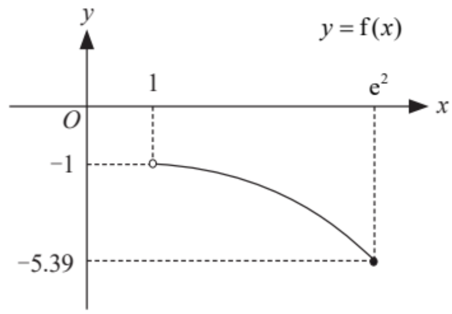


From the graph, the range of h is $R_h = [0, 1]$.

Follow up 6

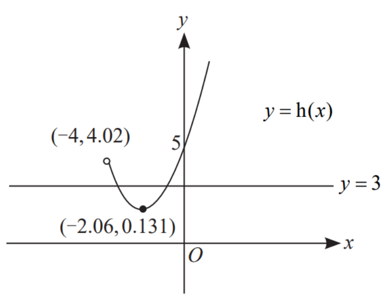
Solution

(a)



For any horizontal line cuts the graph of $y = f(x)$ at most once.
f is a one-one function.

(b)

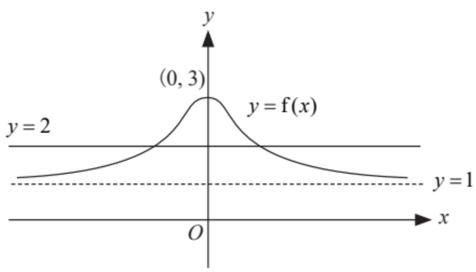


The line $y = 3$ cuts the graph of $y = g(x)$ **more than once**.
g is NOT a one - to - one function.

Follow up 7

Solution

(a) Method 1 (Horizontal Line Test)



The **horizontal line** $y = 2$ that intersects the graph of $y = \mathrm{f}(x)$ **more than once**.
f is **not one - one**, f does not have an inverse.

Method 2 (Counterexample)

Since $\mathrm{f}(-1) = 1 + \dfrac{2}{\mathrm{e}}$ and $\mathrm{f}(1) = 1 + \dfrac{2}{\mathrm{e}}$, f is **not one - one**.
So f does not have an inverse.

(b) Largest value of k is 0.

Follow up 8

Solution

The function is defined by $\mathrm{f} : x \mapsto 2x + 1,\ x \in \mathbb{R}$. Find $\mathrm{f}^{-1}(x)$.

Let
$$y = \mathrm{f}(x)$$
$$y = 2x + 1$$
$$x = \frac{y-1}{2}$$
$$\mathrm{f}^{-1}(y) = \frac{y-1}{2}$$
$$\mathrm{f}^{-1}(x) = \frac{x-1}{2}$$

Follow up 9

Solution

Let $\quad y = \mathrm{f}(x)$

$\qquad = x^2 - 4$

$y + 4 = x^2 - 4x + 4$

$y + 4 = (x-2)^2$

$\pm\sqrt{y+4} = x - 2$

i.e. $\quad x = 2 \pm \sqrt{y+4}$

(a) For $x \geq 2,\ x = 2 + \sqrt{y+4}$

$\mathrm{f}^{-1}(y) = 2 + \sqrt{y+4}$

$\therefore\ \mathrm{f}^{-1}(x) = 2 + \sqrt{x+4}$

(b) For $x \leq 2,\ x = 2 - \sqrt{y+4}$

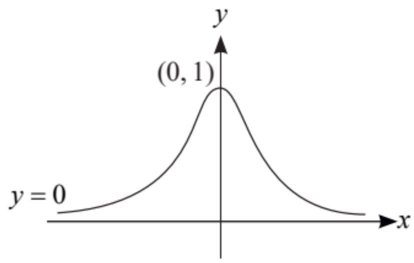
$\mathrm{f}^{-1}(y) = 2 - \sqrt{y+4}$

$\therefore\ \mathrm{f}^{-1}(x) = 2 - \sqrt{x+4}$

Follow up 10

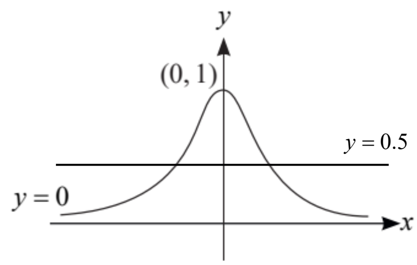
Solution

(a) From the graph, the range of f is $R_f = (0, 1]$.



(b) **Method 1 (Graphical)**

Since the line $y = \frac{1}{2}$ intersects the curve $y = f(x)$ more than once, f is not one - one. (See diagram)



Method 2 (Counterexample)

Since $f(-1) = f(1) = e$, f is not one - one.

(c) The least value of $k = 0$.

Let $y = e^{-x^2}, x \geq 0$

$$\ln y = -x^2$$

$$x^2 = -\ln y$$

$$x = \pm\sqrt{\ln\frac{1}{y}}$$

Since $x \geq 0$, $x = \sqrt{\ln\frac{1}{y}}$

Thus, $f^{-1} : x \mapsto \sqrt{\ln\frac{1}{x}}, x \in \mathbb{R}, 0 < x \leq 1$

(d) To solve $f(x) = f^{-1}(x)$,

consider $f(x) = x$

$$e^{-x^2} = x$$

$$e^{-x^2} - x = 0$$

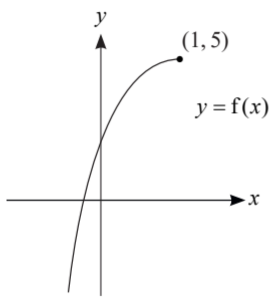
Using GC, $x = 0.653$

Follow up solutions C1 (11-20)

Follow up 11

Solution

(a) The graph of g



Any horizontal line $y = k, k \in \mathbb{R}$, cuts the graph at most once, thus function f is one - one.
Hence function f^{-1} exists.

Let $\quad y = 6 - e^{(x-1)^2}$

$e^{(x-1)^2} = 6 - y$

$(x-1)^2 = \ln(6-y)$

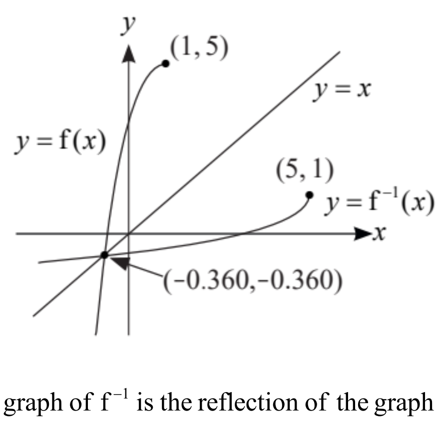
$x = 1 + \sqrt{\ln(6-y)}$ or $x = 1 - \sqrt{\ln(6-y)}$

Since $x \le 1$,

$\therefore\ x = 1 - \sqrt{\ln(6-y)}$

$f^{-1} : x \mapsto 1 - \sqrt{\ln(6-y)}, x \le 5$

(b) The graph of g and g^{-1}



The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Follow up 12

Solution

(a) $\text{fg}(x) = \text{f}(x^2 - 1)$

$= 2(x^2 - 1) + 3$

$= 2x^2 + 1$

(b) $\text{gf}(x) = \text{g}(2x + 3)$

$= (2x + 3)^2 - 1$

$= 4x^2 + 12x + 9 - 1$

$= 4x^2 + 12x + 8$

(c) $\text{ff}^{-1}(x) = \text{f}(\text{f}^{-1}(x))$

$= \text{f}\left(\dfrac{x-3}{2}\right)$

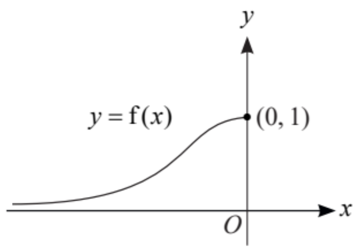
$= 2\left(\dfrac{x-3}{2}\right) + 3$

$= x$

Follow up 13

Solution

(a) The graph of the function f



(b) $R_f = (0, 1]$ (from the graph in **(a)**)

Given $D_g = (0, \infty)$

Since $R_f \subseteq D_g$, thus gf exists.

(c) $\mathrm{gf}(x) = \mathrm{g}\left(\dfrac{1}{x^2+1}\right)$

$$= \frac{1}{\dfrac{1}{x^2+1}} \quad \triangleleft \text{ substitute } \frac{1}{x^2+1} \text{ into } g(x) = \frac{1}{x}.$$

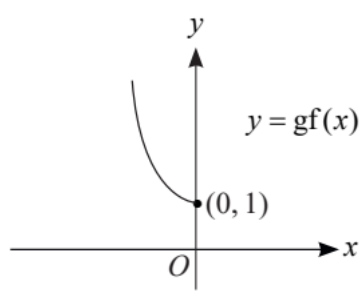
$$= x^2 + 1$$

Since $D_{gf} = D_f = (-\infty, 0]$

$\therefore$ the domain of gf $= (-\infty, 0]$

Using GC to graph $\mathrm{gf}(x) = x^2 + 1$, with domain gf $= (-\infty, 0]$.

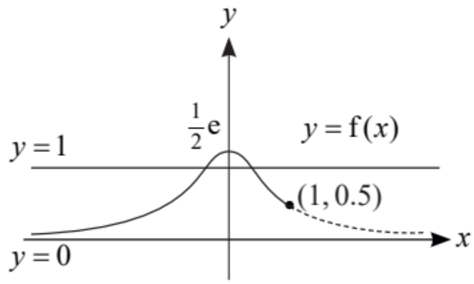
From the graph, $R_{gf} = [1, \infty)$



Follow up 14

Solution

(a) Since the horizontal line $y = 1$ cuts the graph of $y = \mathrm{f}(x)$ twice, f is not a one - one function and thus f^{-1} does not exist.



(b) $b = 0$.

Let $y = \mathrm{f}(x)$

$$y = \frac{1}{2}\mathrm{e}^{1-x^2}$$

$$\ln(2y) = 1 - x^2$$

$$x = \pm\sqrt{1-\ln(2y)}$$

Since $x \le 0$, $\therefore\ x = -\sqrt{1-\ln(2y)}$

$\therefore\ \mathrm{f}^{-1}(x) = -\sqrt{1-\ln(2x)}$

Since $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_\mathrm{f} = \left(0, \frac{1}{2}\mathrm{e}\right]$

The domain of $\mathrm{D}_{\mathrm{f}^{-1}} = \left(0, \frac{1}{2}\mathrm{e}\right]$.

(c) $\mathrm{R}_\mathrm{f} = \left(0, \frac{1}{2}\mathrm{e}\right]$ and $\mathrm{D}_\mathrm{g} = [0, \infty)$.

Since $\mathrm{R}_\mathrm{f} \subseteq \mathrm{D}_\mathrm{g}$, $\therefore$ gf exists.

(d)

$$\begin{aligned} \mathrm{gf}(x) &= \mathrm{g}[\mathrm{f}(x)] \\ &= \mathrm{g}\left[\frac{1}{2}\mathrm{e}^{1-x^2}\right] \\ &= \sqrt{\frac{1}{2}\mathrm{e}^{1-x^2}} \end{aligned}$$

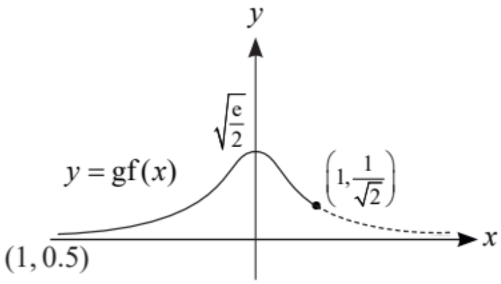
$\therefore\ \mathrm{gf}(x) = \sqrt{\frac{1}{2}\mathrm{e}^{1-x^2}}$

Since $\mathrm{D}_{\mathrm{gf}} = \mathrm{D}_\mathrm{f} = (-\infty, 1]$

Domain of gf $= (-\infty, 1]$

(e) Method 1 (Graphical)

Use GC to graph $\mathrm{gf}(x)=\sqrt{\dfrac{1}{2}\mathrm{e}^{1-x^2}}$, $x \le -1$.

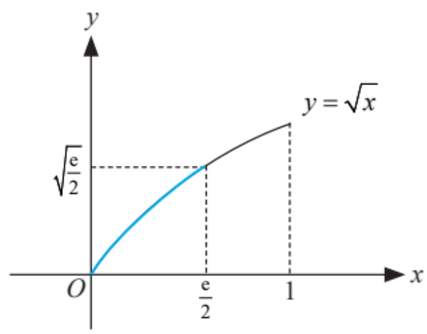


From the graph above, $\mathrm{R_{gf}}=\left(0,\sqrt{\dfrac{\mathrm{e}}{2}}\right]$.

Method 2 (Mapping)

$$\underset{\text{domain f}}{(-\infty,1]} \xrightarrow{\ \mathrm{f}\ } \underset{\text{Range f}}{\left(0,\frac{\mathrm{e}}{2}\right]} \xrightarrow{\ \mathrm{g}\ } \underset{\text{Range gf}}{\left(0,\sqrt{\frac{\mathrm{e}}{2}}\right]}$$

$\therefore \mathrm{R_{gf}}=\left(0,\sqrt{\dfrac{\mathrm{e}}{2}}\right]$ ◃ (Refer to the graph on the right)



Follow up 15

Solution

(a) Let $y=\dfrac{7-3x}{3-x}$ $\triangleleft$ (Expres x in terms of y)

$y(3-x)=7-x$

$x=\dfrac{7-3y}{3-y}$

$\therefore \mathrm{f}^{-1}(x)=\dfrac{7-3x}{3-x}, x\in\mathbb{R}, x\neq 3.$

Since $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$

f is self-inverse (Shown)

In **(a)**, f is self-inverse, $\therefore$ $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$.

$\therefore$ $\mathrm{f}^{n}(x)=x$, if $n=$ even.

$\mathrm{f}^{n}(x)=\mathrm{f}(x)$, if $n=$ odd.

So, $\mathrm{f}^{2003}(x)=\mathrm{f}(x)$.

$\mathrm{f}^{2003}(5)=\mathrm{f}(5)$

$\mathrm{f}^{2003}(5)=\dfrac{7-3(5)}{3-5}$

$=4$

(b) Let $y=\ln(x+1)$

$\mathrm{e}^{y}=x+1$

$x=\mathrm{e}^{y}-1$

$\therefore \mathrm{g}^{-1}(x)=\mathrm{e}^{x}-1$

(c) $\mathrm{R}_{\mathrm{f}^{-1}}=(-\infty,\infty)\setminus\{3\}$ and $\mathrm{D}_{\mathrm{g}}=(-1,2)$

Since $\mathrm{R}_{\mathrm{f}^{-1}}\not\subset \mathrm{D}_{\mathrm{g}}$.

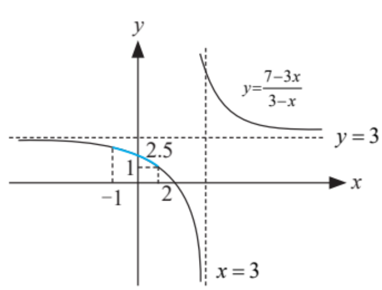
$\therefore$ $\mathrm{g}^{-1}\mathrm{f}$ does not exists.

(d) $\mathrm{R}_{\mathrm{g}^{-1}}=(-1,2)$ and $\mathrm{D}_{\mathrm{f}}=(-\infty,\infty)\setminus\{3\}$

Since $\mathrm{R}_{\mathrm{g}^{-1}}\subseteq \mathrm{D}_{\mathrm{f}}$.

$\therefore$ fg^{-1} exists.

$$\underbrace{(-\infty,\ \ln 3)}_{\mathrm{D}_{\mathrm{g}_{-1}}}\xrightarrow{\mathrm{g}_{-1}}\underbrace{(-1,2)}_{\mathrm{R}_{\mathrm{g}^{-1}}=\text{Restricted }\mathrm{D}_{\mathrm{f}}}\xrightarrow{\mathrm{f}}\underbrace{(1,\ 2.5)}_{\mathrm{R}_{\mathrm{fg}_{-1}}}$$

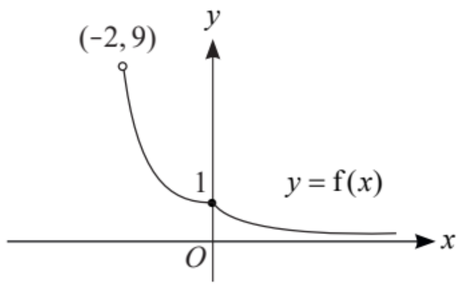


$\therefore \mathrm{R}_{\mathrm{fg}_{-1}}=(1,\ 2.5)$

Follow up 16

Solution

(a) The graph of f



From the graph,
Range of $\mathrm{f} = (0, 9)$

(b) For $-2 < x < 0$,

Let $y = 1 - x^3$

$x = \sqrt[3]{1-y}$

For $x \geq 0$,

Let $y = \mathrm{e}^{-2x}$

$x = -\frac{1}{2}\ln y$

For $x \geq 0$, the range is $0 < \mathrm{f}(x) \leq 1$.

For $-2 < x < 0$, the range is $1 < \mathrm{f}(x) < 9$.

$$\therefore \quad \mathrm{f}^{-1} : x \mapsto \begin{cases} -\frac{1}{2}\ln x, & 0 < x \leq 1, \\ \sqrt[3]{1-x}, & 1 < x < 9. \end{cases}$$

(c) Range of g, $\mathrm{R_g} = [-1, \infty)$ and Domain of f, $\mathrm{D_f} = [-2, \infty)$.

Since $\mathrm{R_g} \subseteq \mathrm{D_f}$,

$\therefore$ fg exists.

(d) $\mathrm{fg}(x) = \mathrm{f}[\mathrm{g}(x)]$

Substitute $x = \alpha$ into $x(x-2)$

Since $\alpha > 2$, $\therefore$ $\alpha(\alpha - 2) > 0$

$\mathrm{fg}(\alpha) = \mathrm{f}(\alpha(\alpha-2)$

$= \mathrm{e}^{-2\alpha(\alpha-2)}$

Given $\mathrm{fg}(\alpha) = \frac{1}{4}$

$\mathrm{e}^{-2\alpha(\alpha-2)} = \frac{1}{4}$

Using GC, $\alpha = 2.30$ (correct to 3 sf)

Follow up 17

Solution

(a)

$f(1) = 3 + 1 = 4$

$$\begin{aligned} f(8) &= f(8-4) \\ &= f(4) \\ &= 7-4 \\ &= 3 \end{aligned}$$

$\therefore\ f(8) = 3$

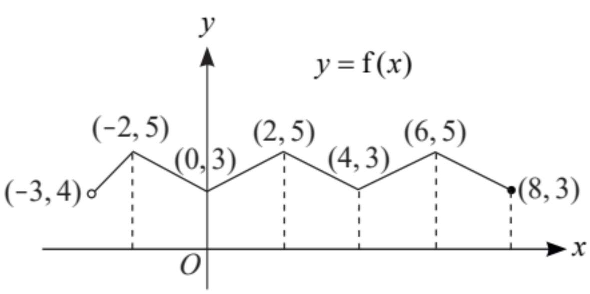
$f(29) = f(25) = f(21)\ \ldots\ldots = f(1)$

So, $f(29) = f(1)$

$$\begin{aligned} &= 3+1 \\ &= 4 \end{aligned}$$

$\therefore\ f(29) = 4$

(b) The graph of f



Learning Point:
The coordinates of the endpoints and the points where the curves meet should be labelled clearly.

(c) Refer to the graph in **(b)**. When $y = 3.5$, the line cuts the graph more than once.
$\therefore$ f is not a 1-1 function.

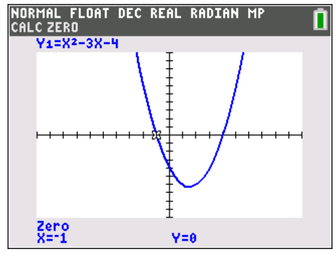
Follow up solutions C2 (1-5)

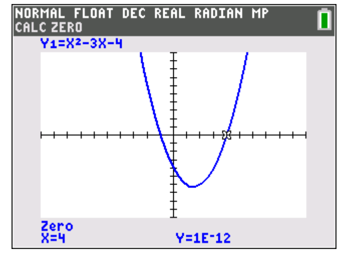
Follow up 1

Solution

Refer to Example 1 in the text $\left(\text{Page } 49\right)$ and follow the steps to find the the x-intercept of the curve

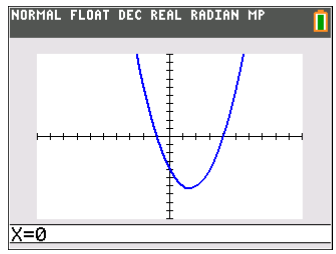
$\therefore$ the x-intercept is -1 and 4.

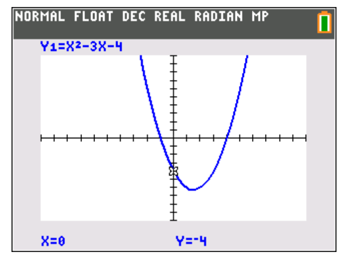




Refer to Example 1 in the text $\left(\text{Page } 49\right)$ and follow the steps to find the the x-intercept of the curve

$\therefore$ the y-intercept is -4.



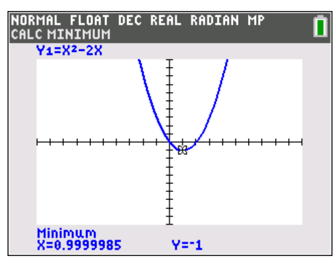


Follow up 2

Solution

Refer to Example 2 in the text $\left(\text{Page } 50\right)$ and follow the steps to find the turning point of the curve.

$\therefore$ the turning point is $(1,-1)$.



Follow up 3

Solution

To find asymptotes, perform long division,

$$y = \frac{2x-3}{x-4}$$

$$= 2 + \frac{5}{x-4}$$

Equations of asymptotes : $y = 2$ (vertical asymptote)

and $x = 4$ (horizontal asymptote)

To determine x-intercept, let $y = 0$.

i.e. $0 = \dfrac{2x-3}{x-4}$

$$x = \frac{3}{2}$$

$\therefore \left(\dfrac{3}{2}, 0\right)$.

To determine y-intercept, let $x = 0$.

i.e. $y = \dfrac{2(0)-3}{0-4}$

$$x = \frac{3}{4}$$

$\therefore \left(0, \dfrac{3}{4}\right)$.

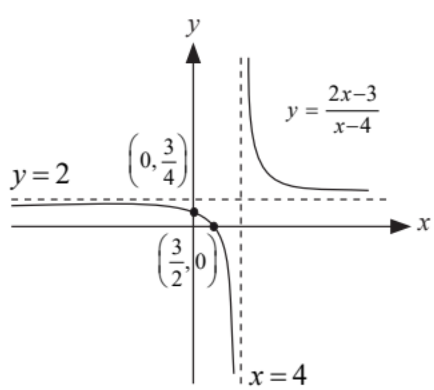
To find stationary point(s)

$$y = 2 + \frac{5}{x-4}$$

$$\frac{dy}{dx} = -\frac{5}{(x-4)^2}$$

For all values of x, $x \neq 4$, $\dfrac{dy}{dx}$ is always negative.

Hence, there are no stationary point on the curve.



Using the above information, we can use GC to graph $y = \dfrac{2x-3}{x-4}$.

Follow up 4

Solution

To find asymptotes, perform long division,

$$y = \frac{x^2 - 3x + 4}{x - 1}$$
$$= x - 2 - \frac{6}{x - 1}$$

Equations of asymptotes : $y = x - 2$ (oblique asymptote) and $x = 1$ (horizontal asymptote)

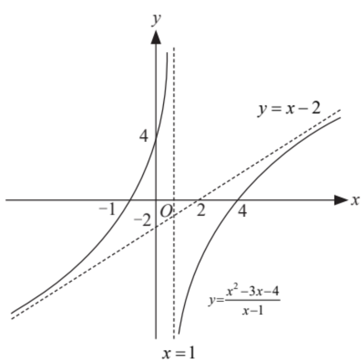
To find stationary points

$$y = x - 2 - \frac{6}{x - 1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{6}{(x - 1)^2}$$

For all values of x, $x \neq 1$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is always positive.

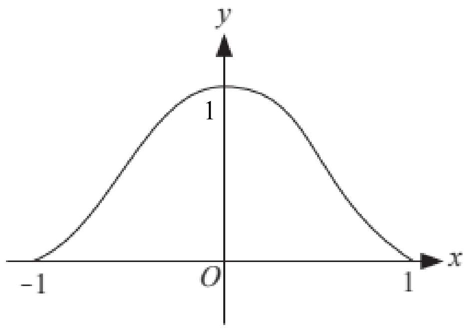
Hence, there are no stationary points on the curve.

Using the above finding, we sketch the graph $y = \frac{x^2 - 3x + 4}{x - 1}$ as shown below.



Follow up 5

Solution



To find the end-points, substitute the value of θ into $x = \cos\theta$, $y = \sin^3\theta$.

When $\theta = 0 : x = 1, \; y = 0$

When $\theta = \dfrac{\pi}{2} : x = 0, \; y = 1$

When $\theta = \pi : x = -1, \; y = 0$

Follow up solutions C2 (6-12)

Follow up 6

Solution

(a) Substitute $t = 0$ into $x = t - 2$.

$\therefore\ x = -2$

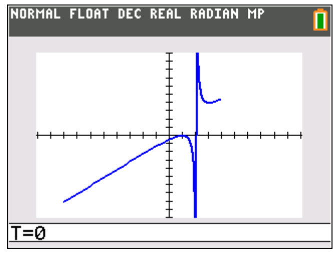
Substitute $t = 0$ into $y = t - 2 + \dfrac{1}{t-4}$.

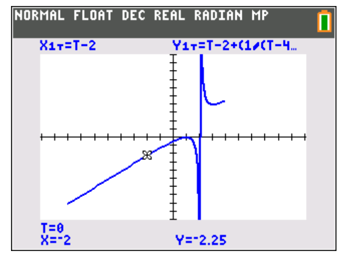
$\therefore\ y = -2.25$

Coordinates $P(-2, -2.25)$

Alternative Method (GC)

We can use GC to find the coordinates of the point P as shown on the right.





(b) Given $x = t - 2$

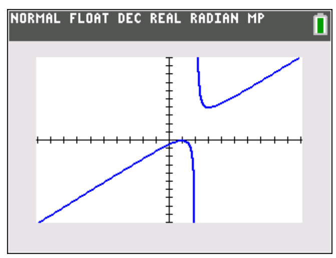
Express $t = x + 2$ .... (1)

Substitute (1) into $y = t - 2 + \dfrac{1}{t-4}$.

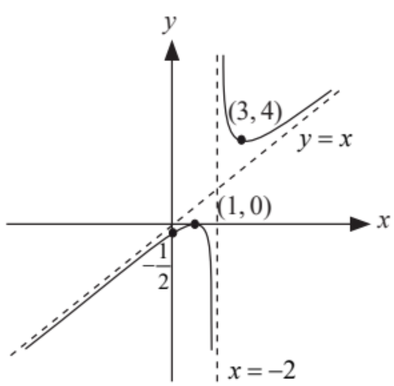
$$\therefore \qquad y = (x+2) - 2 + \frac{1}{(x+2)-4}$$

$$y = x + \frac{1}{x-2}$$

Cartesian equation of C is $y = x + \dfrac{1}{x-2}$



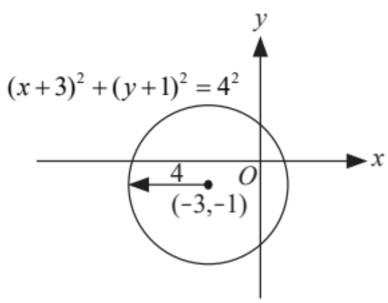
(c)



Follow up 7

Solution

(a)



Circle with centre $(-3,-1)$ and radius 4 units.

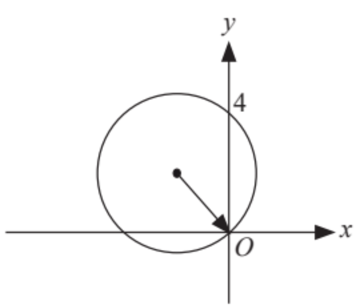
Note:
Lines of symmetry : any straight line passing through the centre.

(b)

$x^2+y^2+2x-4y=0$

$(x^2+2x+1)+(y^2-4y+4)-5=0$ ⊲ (express the x terms and y terms in perfect square form)

$(x+1)^2+(y-2)^2=5$



Circle with centre $(-1,2)$ and radius 5 units.

Follow up 8

Solution

$(4x^2-16x)+(9y^2+18y)=11$

$(4x^2-16x+4)+(9y^2+18y+9)=11$ ◁ (express x terms and y terms in perfect square form)

$4(x^2-4x+4)+9(y^2+2y+1)=1$

$$\frac{4(x-2)^2}{36}+\frac{9(y+1)^2}{36}=\frac{36}{36}$$

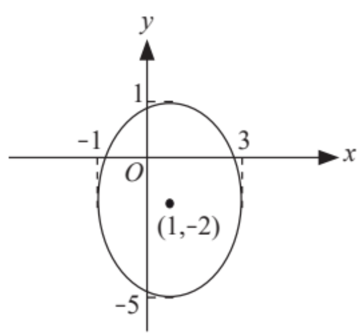
$$\frac{(x-2)^2}{9}+\frac{(y+1)^2}{4}=1$$

$$\frac{(x-2)^2}{3^2}+\frac{(y+1)^2}{2^2}=1$$

Centre of ellipse is $(2,-1)$.

Follow up 9

Solution



$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$

$$\frac{(x-1)^2}{2^2}+\frac{\left(y-(-2)\right)^2}{3^2}=1$$

Ellipse, centre $(1,-2)$.

Note:

Lines of symmetry : $x=1$ and $y=-2$

Follow up 10

Solution

Given $\frac{x^2}{4} - y^2 = 1$

$$\frac{(x-0)^2}{2^2} - \frac{(y-0)^2}{1} = 1$$

$$\frac{x^2}{4} - y^2 = 0$$

$$y^2 = \frac{x^2}{4}$$

$$y = \pm\sqrt{\frac{x^2}{4}}$$

$$y = \pm\frac{x}{2}$$

Equation of the asymptotes are $y = \pm\frac{x}{2}$.

Lines of symmetry of the hyperbola are $x = 0$ and $y = 0$.

Note : The centre of the hyperbola is $(0, 0)$.

Follow up 11

Solution

$x^2-4y^2-2x-8y-39=0$

$(x^2-2x)-(4y+8y)-39=0$ ◁ (completing the square)

$(x-1)^2-1-4(y^2+2y)-39=0$

$(x-1)^2-4\left[(y+1)^2-1\right]-40=0$

$(x-1)^2-4(y+1)^2=36$

$\dfrac{(x-1)^2}{6}-\dfrac{(y+1)^2}{3}=1$ ◁ (Equation of hyperbola in standard form)

Lines of symmetry of the hyperbola are $x=1$ and $y=-1$.

Consider $\dfrac{(x-1)^2}{6^2}-\dfrac{(y+1)^2}{3^2}=0$

$$\frac{(x-1)^2}{6^2}=\frac{(y+1)^2}{3^2}$$

$$(y+1)^2=\frac{(x-1)^2}{2^2}$$

$$(y+1)=\pm\sqrt{\frac{(x-1)^2}{2^2}}$$

$$y=\pm\frac{(x-1)}{2}-1$$

$$y=\frac{x}{2}-\frac{3}{2} \quad \text{or} \quad y=-\frac{x}{2}-\frac{1}{2}$$

$\therefore$ equations of the asymptotes are $y=\dfrac{x}{2}-\dfrac{3}{2}$ and $y=-\dfrac{x}{2}-\dfrac{1}{2}$.

Follow up 12

Solution

Given $36y^2 - 4x^2 = 1$

$$\frac{y^2}{\left(\frac{1}{6}\right)^2} - \frac{x^2}{\left(\frac{1}{2}\right)^2} = 1 \quad \triangleleft \text{ (express in standard form)}$$

(a)

To find y-intercept, let $x = 0$.

$$36y^2 - 4(0)^2 = 1$$
$$y^2 = \frac{1}{36}$$
$$y = \pm\frac{1}{6}$$
$$\therefore \left(0, -\frac{1}{6}\right), \left(0, \frac{1}{6}\right)$$

To find x-intercept, let $y = 0$.

$$36(0)^2 - 4x^2 = 1$$
$$4x^2 + 1 = 0$$

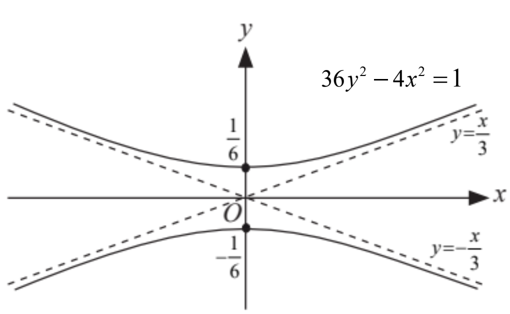
For all values of x, $4x^2 + 1 > 0$.
Hence, there is no x-intercept.

To obtain asymptotes,

consider $36y^2 - 4x^2 = 0$

$$36y^2 = 4x^2$$
$$y^2 = \frac{x^2}{9}$$
$$y = \pm\frac{x}{3}$$

$\therefore$ the equations of the asymptotes are $y = \pm\frac{x}{3}$.

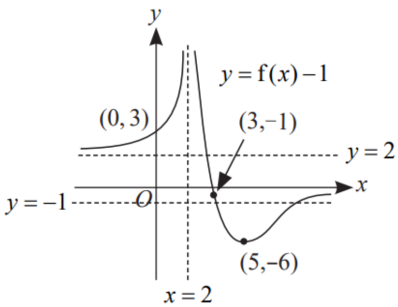


Follow up solutions C3 (1-6)

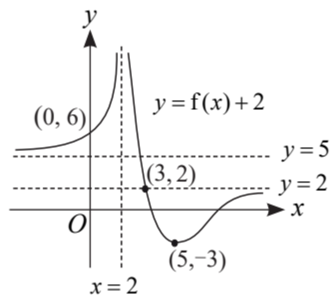
Follow up 1

Solution

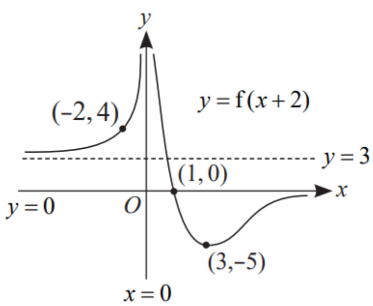
(a)



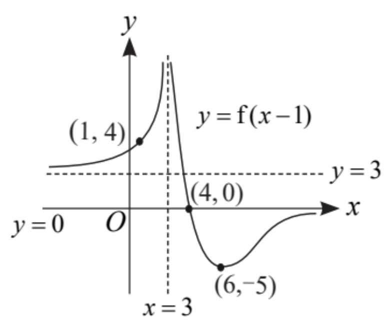
(b)



(c)



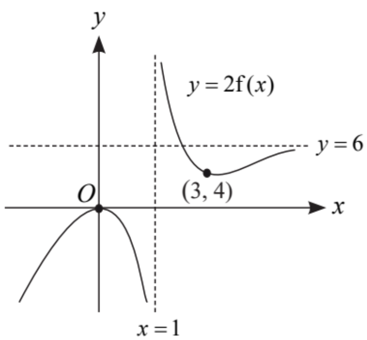
(d)



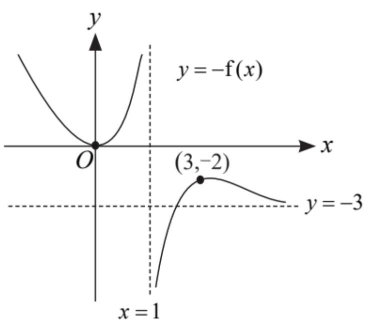
Follow up 2

Solution

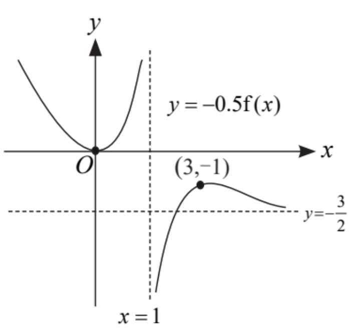
(a)



(b)



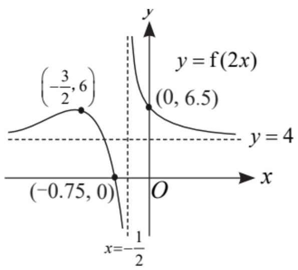
(c)



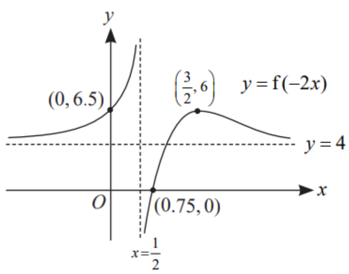
Follow up 3

Solution

(a)



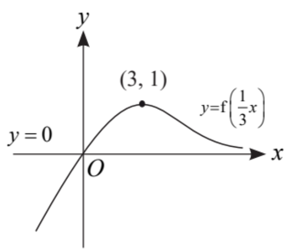
(b)



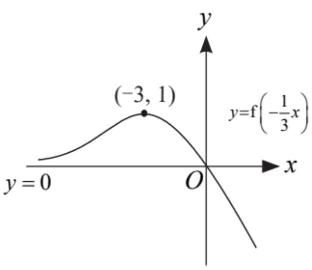
Follow up 4

Solution

(a)

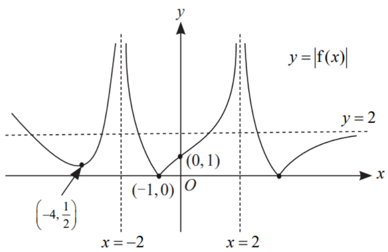


(b)



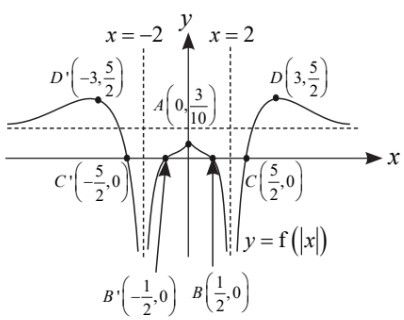
Follow up 5

Solution



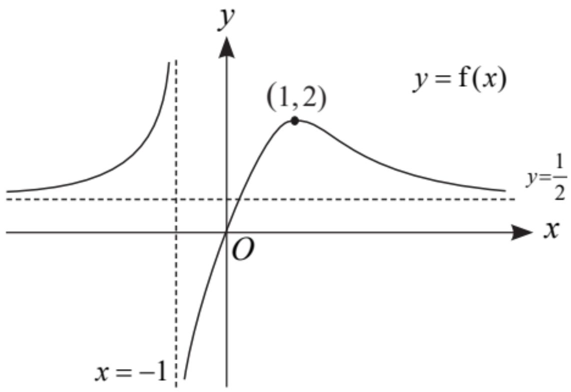
Follow up 6

Solution



Follow up 7

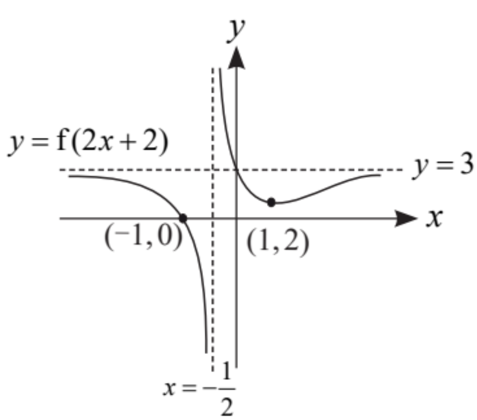
Solution



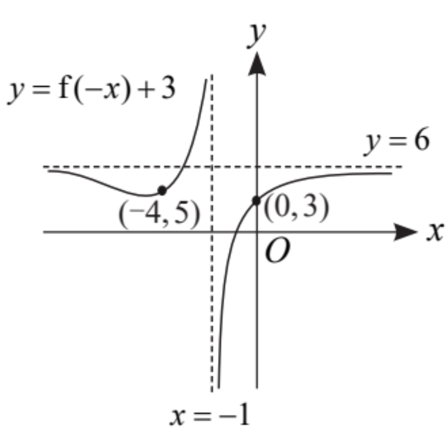
Follow up 8

Solution

(a)

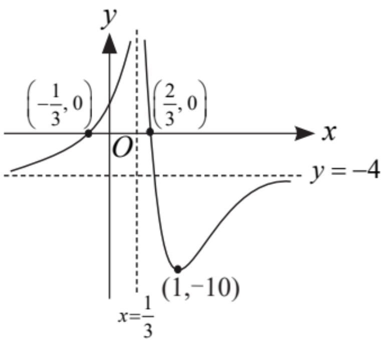


(b)



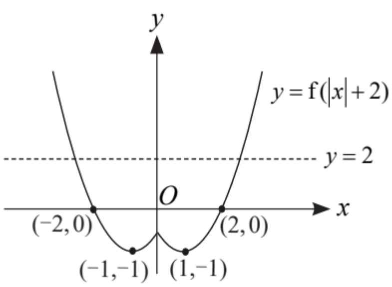
Follow up 9

Solution

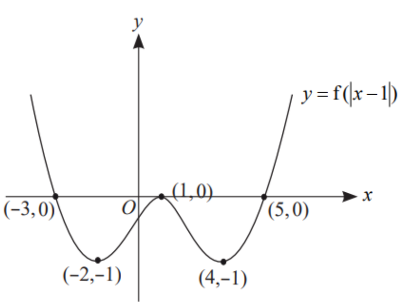


Follow up 10

(a)



(b)



Ejc 2019 midyear j1 /q3check question and check solution from the resource

Follow up 11

The graph of $y = \mathrm{f}(x)$ undergoes the following sequence of transformations.

A: Stretch with scale factor $\frac{2}{3}$ parallel to the x-axis.

B: Reflect about the x-axis

C: Translate 4 units in the negative x-direction

Given that the equation of the resulting curve $y = -\frac{1}{3x+13}$, find the equation of the curve before the 3 transformations were effected.

Solution

Reversing the transformations on $y = -\frac{1}{3x+13}$, we have

$$y = -\frac{1}{3(x-4)+13} \quad \triangleleft \text{ replace } x \text{ by } (x-4), \text{ i.e translation of 4 units in the positive } x\text{-direction}$$

$$y = -\frac{1}{3x+1}$$

$$-y = -\frac{1}{3x+1} \quad \triangleleft \text{ replace } y \text{ by } -y, \text{ i.e. reflection about the } x\text{-axis}$$

$$y = \frac{1}{3x+1}$$

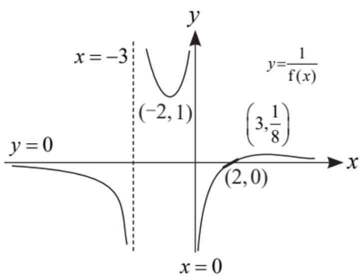
$$y = \frac{1}{3\left(\frac{2}{3}x\right)+1} \quad \triangleleft \text{ replace } x \text{ by } \frac{2}{3}x, \text{ i.e. scaling with scale factor } \frac{3}{2} \text{ parallel to the } x\text{-axis}$$

$$y = \frac{1}{2x+1}$$

$\therefore$ the original equation is $y = \frac{1}{2x+1}$

Follow up 12

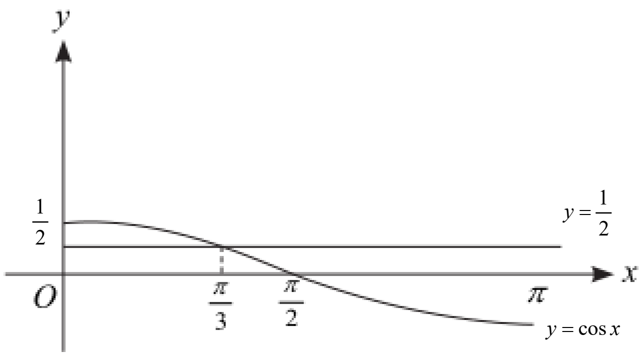
Solution



Follow up solutions C4 (1-13)

Follow up 1

Solution



From the graph, the set of solution is $\{x : x \in \mathbb{R},\ 0 \le x \le \frac{\pi}{3}\}$.

π

Note:

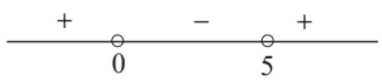
The inequality $\cos x > \frac{1}{2}$ means we are looking for the set of values of such that the graph $y = \cos x$ is above the line $y = 0.5$.

Follow up 2

Solution

(a) $$3x^2 + 10x > 8$$
$$3x^2 + 10x - 8 > 0$$
$$(3x-2)(x+4) > 0$$
$$\therefore \quad x < -4 \text{ or } x > \frac{2}{3}$$

(b) $$x(x-4) < x$$
$$x^2 - 4x - x < 0$$
$$x^2 - 5x < 0$$
$$x(x-5) < 0$$
$$\therefore \quad x > 5 \text{ or } x < 0$$



Follow up 3

Solution

(a)

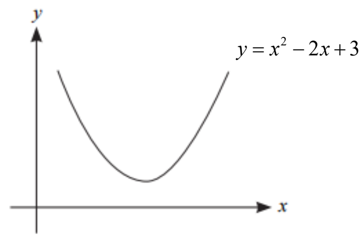
$-x^2 + 2x - 3 < 0$

$x^2 - 2x + 3 > 0$

$(x-1)^2 + 2 > 0$ ◃ By completing the square

$(x-1)^2 + 2$ is always postive, for every values of x.

$\therefore\ x \in \mathbb{R}$



(b) $(x-1)^2 < 2x - 3$

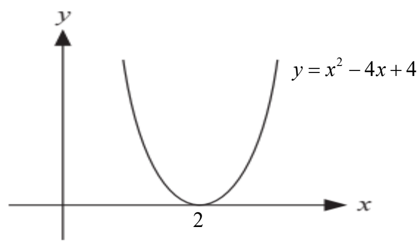
$x^2 - 2x + 1 < 2x - 3$

$x^2 - 4x + 4 < 0$

$(x-2)^2 < 0$

Since $(x-1)^2 + 2 \geq 0$, for every values of x.

x has no solution.

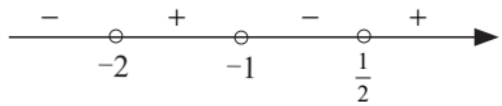


Follow up 4

Solution

Given $(2x-1)(x+2)(x+1)>0$

$\therefore \quad -2<x<-1 \text{ or } x>\frac{1}{2}$



Follow up 5

Solution

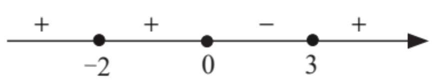
(a)

$x^4+x^3 \le 8x^2+12x$

$x^4+x^3-8x^2-12x \le 0$

$x(x-3)(x+2)^2 \le 0$

$x=-2 \quad \text{or} \quad 0 \le x \le 3$



(b)

$x<0,\ x \ne -2 \text{ or } x>3$

Follow up 6

Solution

(a) Given $\dfrac{x-1}{3x+4}<0$

Hence $-\dfrac{4}{3}<x<1$

(b) Given $x+2\le\dfrac{30}{x+1}$

$$x+2\le\frac{30}{x+1}.\ x\ne-1$$

$$x+2-\frac{30}{x+1}\le 0$$

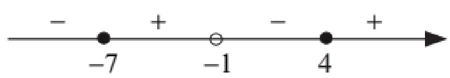
$$\frac{(x+2)(x+1)-30}{x+1}\le 0$$

$$\frac{x^2+x+2x+2-30}{x+1}\le 0$$

$$\frac{x^2+3x-28}{x+1}\le 0$$

$$\frac{(x+7)(x-4)}{x+1}\le 0$$

Hence $x\le -7$ or $-1<x\le 4$



(c) Given $\dfrac{x-1}{x^2}>1$

$x-1>x^2$ ⊲ cross multiply since x^2 is always positve

$x^2-x+1<0$

$$\left(x-\frac{1}{2}\right)^2+\frac{3}{4}<0$$

For all real values of x, $\left(x-\dfrac{1}{2}\right)^2\ge 0$

$\therefore\ \left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}>0.$

$\therefore$ there is no solution of x.

Follow up 7

Solution

To solve $3 < \frac{x}{x-1} \le 2$.

Consider $\quad -3 < \frac{x}{x-1}$ .... (1)

and $\quad \frac{x}{x-1} \le 2$ .... (2)

From (1): $\quad \frac{x}{x-1} > -3$

$$\frac{x}{x-1} + 3 > 0$$

$$\frac{4x-3}{x-1} > 0$$

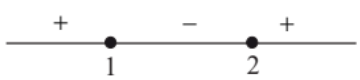
$$x < \frac{3}{4} \text{ or } x > 1 \text{ .... (3)}$$

From (2): $\quad \frac{x}{x-1} \le 2$

$$\frac{x}{x-1} - 2 \le 0$$

$$\frac{x-2(x-1)}{x-1} \le 0$$

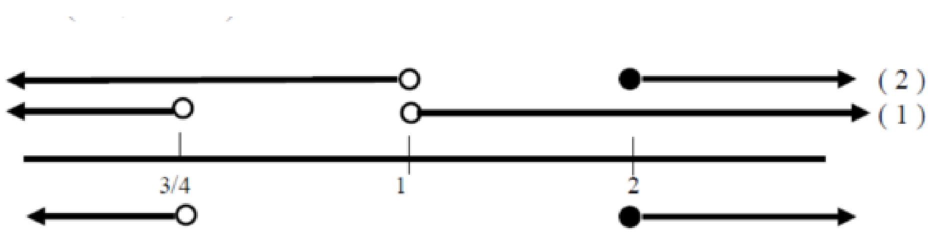
$$\frac{2-x}{x-1} \le 0$$



$$\frac{x-2}{x-1} \ge 0$$

$$x < 1 \text{ or } x \ge 2 \text{ .... (4)}$$

Consider (3) and (4) and draw the number line as shown below (Taking the overlapping lines):



Hence, $\quad x < \frac{3}{4}$ or $x \ge 2$.

Follow up 8

Solution

Given $$\frac{2x^2+5x+9}{x-7} \ge x-4$$

$$\frac{2x^2+5x+9}{x-7}-(x-4) \le 0$$

$$\frac{2x^2+5x+9-(x-4)(x-7)}{x-7} \le 0$$

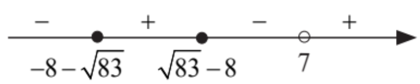
$$\frac{2x^2+5x+9-(x^2-11x+28)}{x-7} \le 0$$

$$\frac{2x^2+5x+9-x^2+11x-28}{x-7} \le 0$$

$$\frac{x^2+16x-19}{x-7} \le 0$$

$$\frac{(x+8)^2-83}{x-7} \le 0$$

$$\frac{(x+8+\sqrt{83})(x+8-\sqrt{83})}{x-7} \le 0$$



$\therefore \quad x \le -8-\sqrt{83}$ or $\sqrt{83}-8 \le x < 7$

Follow up 9

Solution

(a) Given $\dfrac{3x}{x^2-6x+11} > 2$

Since $x^2-6x+11=(x-3)^2+2>0$ for all real x

$\therefore \quad 3x > 2(x^2-6x+11)$ ◃ cross-multiply since $x^2-6x+11$ is always positive

$2x^2-15x+22<0$

$(x-2)(2x-11)<0$

$\therefore \; 2<x<\dfrac{11}{2}$

Altenative Method

$$\frac{3x}{x^2-6x+11}>2$$

$$\frac{3x-2(x^2-6x+11)}{x^2-6x+11}>0$$

$$\frac{2x^2-15x+22}{x^2-6x+11}<0$$

$$\frac{(x-2)(2x-11)}{x^2-6x+11}<0$$

Since $x^2-6x+11=(x-3)^2+2>0$ for all real x

$(x-2)(2x-11)<0$

$\therefore \; 2<x<\dfrac{11}{2}$

(b) $\dfrac{3e^x}{4e^{2x}-12e^x+11}>1$ .... $(*)$

$$\frac{3(2e^x)}{(2e^x)^2-6(2e^x)+11}>2$$

Replace x by $2e^x$ in $(*)$

$\therefore \quad 2<2e^x<\dfrac{11}{2}$

$\quad 1<e^x<\dfrac{11}{4}$

Hence $0<x<\ln\dfrac{11}{4}$

Follow up 10

Solution

(a) Given $|x-2|<3$

$-3<x-2<3$

$-1<x<5$

(b) Given $|x+2|>3$

$x+2>3$ or $x+2<-3$

$x>1$ $\quad$ $x<-5$

$\therefore$ $x>1$ or $x<-5$

(c) Given $-1<|3x+1|<5$

$-1<|3x+1|$ and $|3x+1|<5$

$x\in\mathbb{R}$ $\quad$ $-5<3x+1<5$

$-6<3x<4$

$-2<x<\dfrac{4}{3}$

$\therefore$ $-2<x<\dfrac{4}{3}$

Follow up 11

Solution

(a) Given $\left|2x+1\right|>\left|x-1\right|$

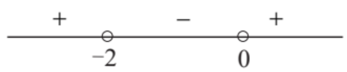
$\left|2x+1\right|^2>\left|x-1\right|^2$ ◁ square both sides

$(2x+1)^2>(x-1)^2$ ◁ $\left|x\right|^2=x^2$

$4x^2+4x+1>x^2-2x+1$

$3x^2+6x>0$

$3x(x+2)>0$



$\therefore\ x<-2$ or $x>0$

(b) Given $\left|\dfrac{x-1}{2x+3}\right|>1$

$\left|x-1\right|>\left|2x+3\right|$

$(x-1)^2>(2x+3)^2$

$x^2-2x+1>4x^2+12x+9$

$0>3x^2+14x+8$

$3x^2+14x+8<0$

i.e. $(3x+2)(x+4)<0$

$-4<x<-\dfrac{2}{3}$

(c) Given $x^2-3|x|-4\geq 0$

$|x|^2-3|x|-4\geq 0$

$(|x|-4)(|x|+1)\geq 0$

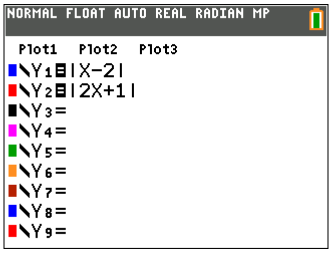
$|x|\geq 4$ or $|x|\leq -1$ (No Solution)

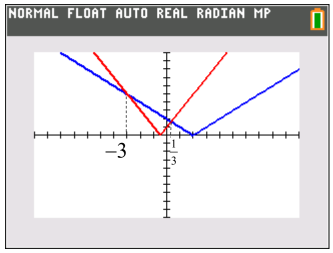
$x\geq 4$ or $x\leq -4$

Follow up 12

Solution

Sketch the graphs of $y = |x-2|$ and $y = |2x+1|$.





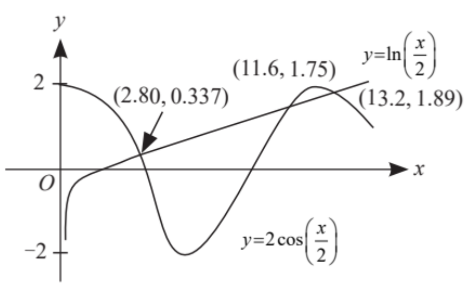
The x-coordinates of 2 intersection points are $x = -3$ and $x = \frac{1}{3}$ respectively.

From the graph, for $|x-2| \le |2x+1|$

$\therefore \quad x \le -3 \text{ or } x \ge \frac{1}{3}$

Follow up 13

Solution



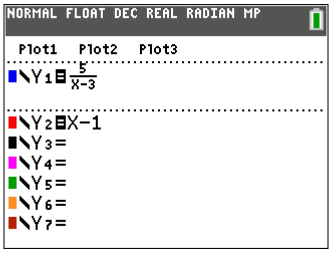
From the graph, $0 < x \le 2.80$ or $11.6 \le x \le 13.2$

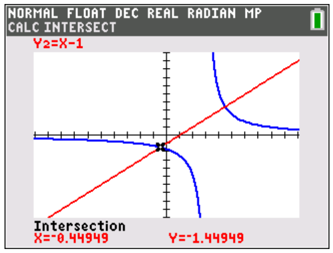
Follow up solutions C2 (14-20)

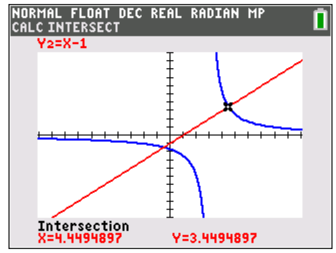
Follow up 14

Solution

To solve the equation $\dfrac{5}{x-3} = x-1$, we use GC to find the intersection between the two graphs as shown.





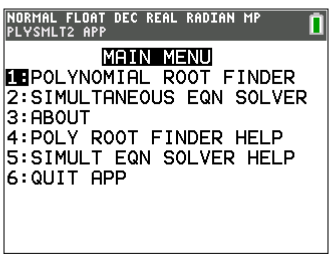


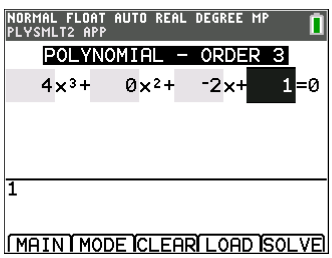
From GC, $x = -0.45$ or 4.45

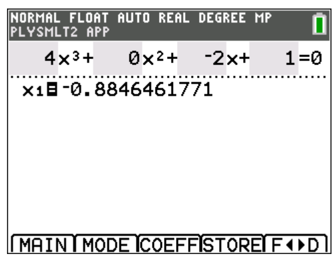
Follow up 15

Solution

To solve a polynomial equation (such as $4x^3 - 2x + 1 = 0$), we can use polynomial finder as shown.







From GC, $x = -0.88$ (correct to 2 dp)

Follow up 16

Solution

$$a+c=4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$-3a+b+c=2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$9a+2b+c=16 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Using GC, $a=1,\ b=2,\ c=3$

Follow up 17

Solution

Let the equation of the cubic curve be $y=ax^3+bx^2+cx+d$.
Given that the y-intercept is -4, i.e. when $x=0,\ y=4$.
$\therefore\ d=-4$
The cubic curve passes through the points $(1,-3)$, i.e. $x=1,\ y=-3$.
Substitute $x=1,\ y=-3$ into $y=ax^3+bx^2+cx+d$

$$-3=a+b+c-4$$
$$1=a+b+c \ldots\ldots\ldots\ldots\ldots (1)$$

The cubic curve passes through the points $(2,12)$, i.e. $x=2,\ y=12$.
Substitute $x=2,\ y=12$ into $y=ax^3+bx^2+cx+d$

$$12=8a+4b+2c-4$$
$$16=8a+4b+2c \ldots\ldots\ldots\ldots\ldots (2)$$

The cubic curve passes through the points $(3,59)$, i.e. $x=3,\ y=59$.

$$59=27a+9b+3c-4$$
$$63=27a+9b+3c$$

Use GC to solve (1), (2) and (3): $a=3,\ b=-2,\ c=0$

$\therefore\ y=3x^3-2x^2-4$

Follow up 18

Solution

Let $x,\ y$ and z be the number of packs of chicken, mutton and beef that Mrs Tan have to buy.
$7.45x+4.60y+8.95z=89.70$ ...(1)
$7.20x+4.40y+8.70z=86.80$ ...(2)
$7.15x+4.45y+8.90z=87.40$ ...(3)
Using GC, $x=12,\ y=1$ and $z=4$
Mrs Tan has to buy 12kg of chicken, 1kg of mutton and 4kg of beef.

Follow up 19

Solution

Let $x,\ y$ and z be the number of successful spins that won 40, 60 or 100 coins respectively.

$$x+y+z=2395\times 0.8$$

$$40x+60y+100z=117640$$

$$(0.25\times 40x)+(0.35\times 60y)+(0.40\times 100z)=40255$$

From G.C.

$x=836,\ y=595,\ z=485.$

Hence, there were 485 spins won 100 coins that month.

Follow up 20

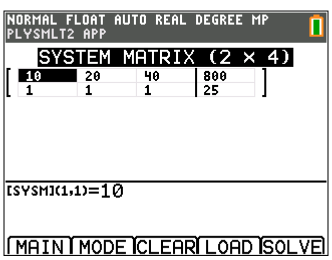
Solution

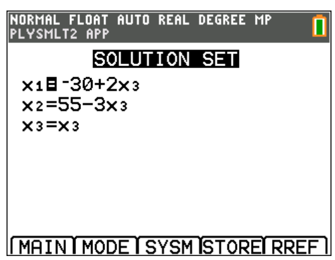
(a) If x, y and z represents the number of 4.2-metre, 5.3-metre and 6.4-metre trucks leased respectively, we set up the following pair of equations with the information provided.

$$10x+20y+40z=800$$

$$x+y+z=25$$

Using GC





Solving in terms of z (see diagram 2) we obtain

$x=-30+2z$ .... (1)

$y=55-3z$ .... (2)

Since x, y and z are all non-regatime integeas

From (1): $2z-30\geq 0$

$z\geq 15$

From (2): $55-3z\geq 0$

$z\leq 18$

hence, $15\leq z\leq 18$

Substitute these four values ($z=15,16,17,18$) into (1) and (2) to get the corresporeding x and y values.

The possible combinations of the number of each type of truck that the company could lease are (0, 10, 15), (2, 7, 16), (4, 4, 17) and (6, 1, 18).

(b) Calculate the total rental charges for each type of truck, as shown in the table.

Possible combination (x, y, z)	Total rental charge : $115x + 150y + 200z$
(0, 10, 15)	4500
(2, 7, 16)	4480
(4, 4, 17)	4460
(6, 1, 18)	4440

For the company to achieve minimum daily leasing cost, the company needs to lease 6 of 4.2-metre trucks, 1 of 5.3-metre truck and 18 of 6.4-metre trucks.

Follow up solutions C2 (1-5)

Follow up 1

Solution

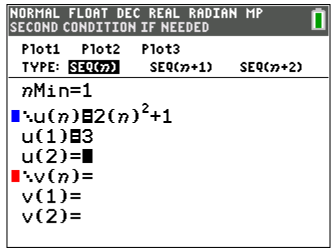
Let $u_n = 2n^2 + 1$ .... (1)

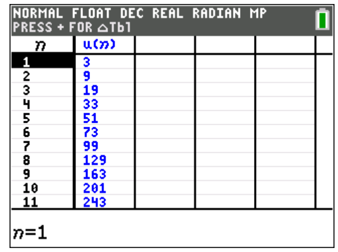
Substituting $n = 1$ into (1) gives $u_1 = 3$.

Substituting $n = 2$ into (1) gives $u_2 = 9$.

Substituting $n = 3$ into (1) gives $u_3 = 19$.

Substituting $n = 4$ into (1) gives $u_4 = 33$.





Alternatively, we can use GC to determine the values.

Follow up 2

Solution

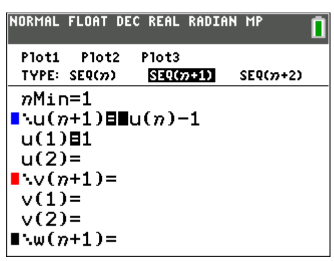
Given $u_1 = 1$ and $u_{n+1} = 3u_n - 1$.

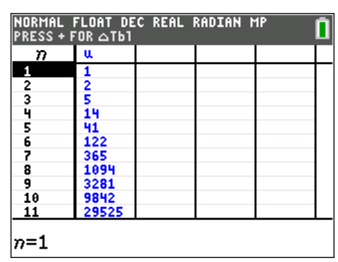
$$\begin{aligned} u_2 &= 3u_1 - 1 \quad \triangleleft\ u_1 = 1 \\ &= 3(1) - 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} u_3 &= 3u_2 - 1 \quad \triangleleft\ u_2 = 2 \\ &= 3(2) - 1 \\ &= 5 \end{aligned}$$

$$\begin{aligned} u_4 &= 3u_3 - 1 \quad \triangleleft\ u_3 = 5 \\ &= 3(5) - 1 \\ &= 14 \end{aligned}$$

Alternatively, we can use GC to determine the values.





Follow up 3

Solution

(a) $x_n = 0.9x_{n-1} + 90, n = 1, 2, 3...$

(b)
$$\begin{aligned}
x_n &= 0.9x_{n-1} + 90 \\
&= 0.9(0.9x_{n-2} + 90) + 90 \\
&= 0.9^2 x_{n-2} + 0.9(90) + 90 \\
&= 0.9^2 (0.9x_{n-3} + 90) + 0.9(90) + 90 \\
&= 0.9^3 x_{n-3} + 0.9^2 (90) + 0.9(90) + 90 \\
&= \ldots \\
&= 0.9^n x_0 + 0.9^{n-1}(90) + \ldots + 0.9^2 (90) + 0.9(90) + 90 \\
&= 0.9^n x_0 + 90\left(\frac{1-0.9^n}{1-0.9}\right) \\
&= 0.9^n x_0 + 900(1-0.9^n) \\
&= 0.9^n (x_0 - 900) + 900 \quad \text{(Shown)}
\end{aligned}$$

Follow up 4

Solution

$$\begin{aligned}
u_n &= S_n - S_{n-1} \\
&= (2^n - 1) - (2^{n-1} - 1) \\
&= 2^n - 2^{n-1} \\
&= 2^{n-1}(2-1) \\
&= 2^{n-1}
\end{aligned}$$

nth term of the series $u_n = 2^{n-1}$.

Follow up 5

Solution

(a)

$$
\begin{aligned}
u_n &= S_n - S_{n-1} \\
&= n(2n+c) - (n-1)(2n-2+c) \\
&= 2n^2 + cn - (2n^2 - 2n + cn - 2n + 2 - c) \\
&= 4n - 2 + c
\end{aligned}
$$

(b)

$$
\begin{aligned}
u_n &= 4n - 2 + c \qquad \triangleleft \text{ (replace } n \text{ by } n+1) \\
u_{n+1} &= 4(n+1) - 2 + c \\
&= 4n + 4 - 2 + c \\
&= 4n - 2 + c + 4 \\
&= u_n + 4 \quad \text{(Shown), where } k = 4
\end{aligned}
$$

Follow up solutions C5 (6-13) Arithmetic

Follow up 6

Solution

Take $-5-2=-7$
and $-12-(-5)=-7$

$\therefore$ the common difference $=-7$

Adding -7 to each of the next two term,
$\therefore$ the next two terms of the sequence are -19 and -26.

Follow up 7

Solution

nth term formula for AP: $u_n = a+(n-1)d$
First term, $a=2$
common difference, $d=7$
Substitute $a=2$ and $d=7$ into $u_n = a+(n-1)d$.

$$\begin{aligned} u_n &= 2+(n-1)\times 9 \\ &= 9n-7 \end{aligned}$$

Substitute $n=50$ into $u_n = 9n-7$.

$$\begin{aligned} \therefore \quad u_{50} &= 9\times 50-7 \\ &= 443 \end{aligned}$$

Follow up 8

Solution

Let u_n be the general term of arithmetic progression.
Formula : $u_n = a + (n-1)d$

Given $u_4 = 7$ $\triangleleft$ 4th term $= u_4$

$a + 3d = 7$ .... (1)

Also given

$u_{10} = 16$ $\triangleleft$ 10th term $= u_{10}$

$a + 9d = 16$ .... (2)

$(2)-(1)$ gives

$6d = 9$

$d = 1.5$

Substituting into $d = 1.5$ to (1) gives

$a + 4.5 = 7$

$a = 2.5$

First term $= 2.5$, common difference $= 1.5$.

Follow up 9

Solution

Let T_n be the general term of arithmetic progression.
From the given information: The largest piece of wood is 5 times the length of the smallest piece of wood.

i.e $T_7 = 5T_1$

$a + 6d = 5a$ $\triangleleft$ (Formula : $u_n = a + (n-1)d$)

$6d = 4a$

$\frac{3}{2}d = a$ .... (1)

Also given information: The length of third piece of wood is 17.5 cm.

i.e $T_3 = 17.5$

$a + 2d = 17.5$

$\frac{3}{2}d + 2d = 17.5$

$3\frac{1}{2}d = 17.5$

$d = 5$ .... (2)

Substituting (2) into (1) gives $a = \frac{15}{2}$.

The length of the smallest piece is $\frac{15}{2}$ cm.

Follow up 10

Solution

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= n^2 + 3n - \left[(n-1)^2 + 3(n-1)\right] \\ &= n^2 + 3n - \left[n^2 + n - 2\right] \\ &= 2n + 2 \end{aligned}$$

$\therefore$ nth term of the series is $2n+2$.

$$\begin{aligned} u_{n+1} &= 2(n+1) + 2 \\ &= 2n + 4 \end{aligned}$$

$u_{n+1} - u_n$
$= (2n+4) - 2n - 2$
$= 2$

Since the difference between any two consecutive terms is the same, the sequence is an A.P.

Follow up 11

Solution

(a) In the series, $a = 4$, $d = -7$ and $n = 35$.

Using the sum formula of AP: $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$$\begin{aligned} S_{35} &= \frac{35}{2}\left[2(18) + 34(-7)\right] \\ &= -3535 \end{aligned}$$

(b) In the series, $a = 152$, $d = -3$ and $l = u_n = 50$.

Using the nth formula of AP, $u_n = a + (n-1)d$ to find n.

$$\begin{aligned} 50 &= 152 + (n-1)(-3) \\ -102 &= (n-1)(-3) \\ 34 &= n - 1 \\ \therefore \quad n &= 35 \end{aligned}$$

Use $S_n = \frac{n}{2}(a + l)$

$$\begin{aligned} S_{35} &= \frac{35}{2}(152 + 50) \\ &= \frac{35}{2}(152 + 50) \\ &= 3535 \end{aligned}$$

Follow up 12

Solution

(a) Use $u_n = a + (n-1)d$, where $a = 10$, $u_{100} = 1$, $n = 100$

$$u_{100} = 10 + (100-1)d$$
$$1 = 10 + 99d$$
$$d = -\frac{1}{11}$$

$\therefore$ the common difference is 4

Use $S_n = \frac{n}{2}[2a + (n-1)d]$, where $a = 10$, $d = -\frac{1}{11}$, $n = 50$

$$S_{50} = \frac{50}{2}\left[2(10) + (50-1)\left(-\frac{1}{11}\right)\right]$$
$$= \frac{4275}{11}$$

$\therefore$ the sum of this fifty terms of this series is $\frac{4275}{11}$

(b) **Solution**

Use $S_n = \frac{n}{2}[2a + (n-1)d]$, where $S_{18} = 234$, $d = -0.2$, $n = 18$

$$234 = \frac{18}{2}[2a + (18-1)(-0.2)]$$
$$26 = 2a - 3.4$$
$$29.4 = 2a$$
$$a = 14.7$$

$\therefore$ the first term is 14.7

Follow up 13

Solution

Given $\quad T_{12} = 52$

$$a + 11d = 52 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1)$$

and

$$S_{18} = 756$$

$$\frac{18}{2}(2a + 17d) = 756$$

$$2a + 17d = 84 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2)$$

Solving (1) and (2) gives

$$5d = 20$$

$$d = 4$$

Substituting $d = 4$ into (1) gives

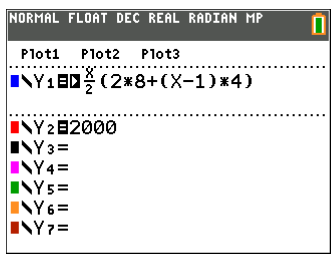
$$a + 11(4) = 52$$

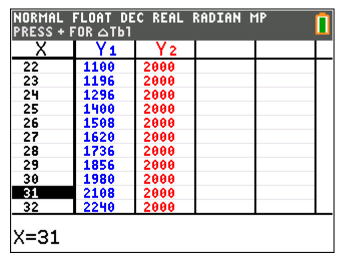
$$a = 8$$

To find $\quad S_n > 2000$

$$\frac{n}{2}[2(8) + (n-1)4] > 2000$$

Using GC, the least n is 31.





Follow up solutions C5 (14-22) Geometric

Follow up 14

Solution

Common ratio, $r = \frac{21}{7}$

$= 3$

nth term formula for GP: $u_n = ar^{n-1}$

First term, $a = 7$

Common difference, $r = 3$

Term, $n = 8$

Substitute $a = 2,\ , n = 8$ and $d = 7$ into $u_n = ar^{n-1}$.

$$\therefore \quad u_8 = ar^{n-1}$$
$$= 7(3)^{8-1}$$
$$= 15309$$

8th term of this geometric progression is 15309.

Follow up 15

Solution

(a) $S_n = 1 - \left(k - \frac{3}{4}\right)^n$

Replace n by $n-1$, $S_{n-1} = 1 - \left(k - \frac{3}{4}\right)^{n-1}$

nth term $= S_n - S_{n-1}$

$$= \left[1 - \left(k - \frac{3}{4}\right)^n\right] - \left[1 - \left(k - \frac{3}{4}\right)^{n-1}\right]$$

$$= \left(k - \frac{3}{4}\right)^{n-1} - \left(k - \frac{3}{4}\right)^n$$

$$= \left(k - \frac{3}{4}\right)^{n-1}\left[1 - \left(k - \frac{3}{4}\right)\right]$$

$$= \left(k - \frac{3}{4}\right)^{n-1}\left(\frac{7}{4} - k\right)$$

(b) $$\frac{U_{n+1}}{U_n} = \frac{\left(k - \frac{3}{4}\right)^n\left(\frac{7}{4} - k\right)}{\left(k - \frac{3}{4}\right)^{n-1}\left(\frac{7}{4} - k\right)}$$

$$= \frac{1}{\left(k - \frac{3}{4}\right)^{-1}}$$

$$= k - \frac{3}{4}$$

$k - \frac{3}{4}$ is constant since k is a constant.

Hence, the series is in geometric progression.

Follow up 16

Solution

Solution

Given $u_2 = 12$ ⊲ Use nth formula of GP, $u_n = ar^{n-1}$, where $n = 2$

i.e. $ar = 12$ .... (1)

Also $u_4 = 27$ ⊲ Use nth formula of GP, $u_n = ar^{n-1}$, where $n = 4$

$ar^3 = 27$..(2)

Taking $\dfrac{(2)}{(1)}$

$$r^2 = \frac{27}{12}$$

$r = 1.5$ or $r = -1.5$ (Rejected since all the terms are positive)

$\therefore$ the common ratio is 1.5.

Substituting $r = 1.5$ into (1) to find a.

$\therefore$ $a(1.5) = 12$

$$a = \frac{12}{(1.5)}$$

$$a = 8$$

$\therefore$ the first term is 8.

Follow up 17

Solution

(a) From the given information: First term, $a=3$ and common ratio, $r=\dfrac{2}{3}$.

$$S_8=\frac{3\left[1-\left(\frac{2}{3}\right)^8\right]}{1-\frac{2}{3}}$$

$$=8.65$$

(b) Given that the third term of a geometric progression is nine times the first term.

i.e. $\quad$ 3rd term $=9\times$ first term

$$ar^2=9a$$

$$r=\pm 3$$

Also given that the sum of the first six terms is k times the sum of the first two terms.

i.e. $\quad S_6=kS_2$

$$\frac{a(r^6-1)}{r-1}=\frac{ka(r^2-1)}{r-1}$$

$$k=\frac{r^6-1}{r^2-1}$$

Substitute $r=3$ into $k=\dfrac{r^6-1}{r^2-1}$.

$\therefore \quad k=91$

Substitute $r=-3$ into $k=\dfrac{r^6-1}{r^2-1}$.

$\therefore \quad k=91$

Hence, $k=91$.

Follow up 18

Solution

(a) To prove the sequence is GP, we need to show $\frac{v_n}{v_{n-1}}$ is a constant.

$$\frac{v_n}{v_{n-1}} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n}{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1}}$$

$$= \frac{2}{3}$$

Since $\frac{v_n}{v_{n-1}} = \frac{2}{3}$ which is a number, the sequence is a GP. (Shown)

(b) Given $v_n = \frac{1}{2}\left(\frac{2}{3}\right)^n$ ◁ substitute $n = 1$

$$v_1 = \frac{1}{2}\left(\frac{2}{3}\right)^1$$

$$= \frac{1}{3}$$

$\therefore$ the first term is $\frac{1}{3}$.

Common ratio $= \frac{2}{3}$ (from **(a)**)

$v_1,\ v_2,\ v_3 + \ldots + v_{2n}$ ◁ sum of first $2n$ terms

$$= \frac{\frac{1}{3}\left(1 - \left(\frac{2}{3}\right)^{2n}\right)}{1 - \frac{2}{3}}$$ ◁ use sum GP formula: $S_n = \frac{a(1-r^n)}{1-r}$, where $n = 2n,\ r = \frac{2}{3}$ and $a = \frac{1}{3}$

$$= 1 - \left(\frac{2}{3}\right)^{2n}$$

$$= 1 - \left(\frac{4}{9}\right)^n$$ (Shown)

Follow up 19

Solution

Let the areas of the sectors be $A_1, A_2, \ldots\ldots A_{22}$.

Given that the area of the eighth sector is twice the area of the smallest sector

i.e. $A_8 = 2A_1$

$$A_1 + 7d = 2A_1$$

$$A_1 = 7d \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Sum of all the areas of sectors = Area of the circle

$$\frac{22}{2}[2A_1 + 21d] = \pi r^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

Substitute (1) into (2)

$$\frac{22}{2}[14d + 21d] = \pi r^2$$

$$11[35d] = \pi r^2$$

$$d = \frac{1}{385}\pi r^2$$

Let the angle of the largest sector be θ.

Area of the largest sector sector, $A_{22} = \frac{1}{2}r^2\theta$

$$A_{22} = A_1 + 21d \qquad \triangleleft \text{ In (1), since } A_1 = 7d)$$

$$A_{22} = 7d + 21d$$

$$= 28d$$

$$\frac{1}{2}r^2\theta = 28\left(\frac{1}{385}\pi r^2\right)$$

$$\theta = \frac{8\pi}{55}$$

Follow up 20

Solution

(a) From the given information: $a = 10000$, $r = 1.1$ and $n = 7$.

$$U_7 = (10000)(1.1)^{7-1}$$
$$= 17715.61$$

The value of the donation in 2016 is \$17 715.61.

(b) From the years 2010 to 2016, inclusive, there are 7 years, i.e. $n = 7$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$ ◁ use sum of AP formula

$$= \frac{10000(1.1^7 - 1)}{1.1 - 1}$$
$$= 94871.71$$

Total value of the donations made during the years 2010 to 2016, inclusive is \$94 871.71.

Follow up 21

Solution

(a) Let A_n be the distance travelled by Alex on Day n.

$A_{10} = 1000 + (10-1)(-5)$
$= 955 \text{ km}$

(b) Let G_n be the distance travelled by Gopal on Day n.

$G_5 = 1500, G_6 = 1500(0.98), G_7 = 1500(0.98)^2, \ldots, G_n = 1500(0.98)^{n-5}$

$G_{10} = 1500(0.98)^{10-5}$
$= 1355.88 \text{ km}$
$\approx 1356 \text{ km}$ (Correct to nearest whole number) (Shown)

(c) Total distance travelled by Alex by the end of Day 100

$= \dfrac{100}{2}[2(1000) + (100-1)(-5)]$
$= 75250 \text{ km}$

Total distance travelled by Gopal

$1500 + 1500(0.98) + \ldots + 1500(0.98)^{n-5} + \ldots$
$= \dfrac{1500}{1-0.98}$
$= 7500 \text{ km} < 75250 \text{ km}$

Hence, Gopal is unable to complete the journey.

(d) Total distance travelled by Gopal at the end of Day n

$= 1500 + 1500(0.98) + \ldots + 1500(0.98)^{n-5}$
$= \dfrac{1500(1-0.98^{n-4})}{1-0.98}$

Total distance travelled by Alex at the end of Day $n-1$

$= 1000 + 995 + \ldots + \left[1000 + (n-1-1)(-5)\right]$
$= \dfrac{n-1}{2}[2(1000) + (n-2)(-5)]$

Given that the total distance travelled by Gopal at the end of Day n exceeds the total distance travelled by Alex at the end of Day $n-1$

i.e. $\dfrac{n-1}{2}[2(1000) + (n-2)(-5)] < \dfrac{1500(1-0.98^{n-4})}{1-0.98}$

$\dfrac{n-1}{2}[2(1000) + (n-2)(-5)] - \dfrac{1500(1-0.98^{n-4})}{1-0.98} < 0$

n	$\frac{n-1}{2}[2(1000)+(n-2)(-5)]-\frac{1500(1-0.98^{n-4})}{1-0.98}$
10	258.18
11	−115.59
12	−467.77

Using G.C., the least $n = 11$. (See table)

Total distance travelled by Gopal by the end of Day 11

$= \dfrac{1500(1-0.98^{11-4})}{1-0.98}$
$= 9890.585 \text{ km}$
$= 9891$ (Correct to nearest whole number)

Follow up 22

Solution

(a)

Year	Amt at beginning of year
(1)	1000
(2)	$1000(0.04)-50$
(3)	$[1000(1.04)-50](1.04)-50$
	$=1000(1.04)^2-50(1.04)^1-50(1.04)^0$
(4)	$[1000(1.04)^2-50(1.04)-50](1.04)-50$
	$=1000(1.04)^3-50(1.04)^2-50(1.04)^1-50(1.04)^0$

By observation:

For $(n+1)$ year $\quad 1000(1.04)^n-50(1.04)^{n-1}-\ldots-50(1.04)^1-50(1.04)^0$

Amount at beginning of the $(n+1)$th year :

$=1000(1.04)^n-50(1.04)^{n-1}-\ldots-50(1.04)-50(1.04)^0$

$=1000(1.04)^n-50\left[(1.04)^{n-1}+\ldots+(1.04)+(1.04)^0\right]$

$=1000(1.04)^n-50\left[(1.04)^0+(1.04)+\ldots+(1.04)^{n-1}\right]$

$=1000(1.04)^n-50\left[\dfrac{(1)(1.04^n-1)}{(1.04-1)}\right]$

$=1250-250\ (1.04)^n$

$=1250\left[1-\dfrac{1}{5}(1.04)^n\right]$ .... (1) (Shown)

(b)

After 20 years, i.e. $n=20$. Substituting $n=20$ into (1)

Amount at the end of the 20th year

$=1250\left[1-\dfrac{1}{5}(1.04)^{20}\right]$

$=702.2$

$\therefore$ the amount the Caitlin has after twenty years is \$703.

Let the number of years to be lasted be k.

i.e. $1250\left[1-\dfrac{1}{5}(1.04)^k\right]=0 \quad \triangleleft$ substitute $n=k$ into $1250\left[1-\dfrac{1}{5}(1.04)^n\right]$

$k=41.035$

$\therefore$ it will last for 41 years.

Follow up 23

Solution

(a)

As $n \to \infty,\ n^2 + 1 \to \infty$

$\therefore$ the sequence u_n diverges and the limit of the sequence does not exist.

(b)

$$\lim_{n\to\infty}\left(\frac{-4n^2+3n-2}{n^2+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{-4n^2}{n^2}+\frac{3n}{n^2}-\frac{2}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{-4-\frac{3}{n}-\frac{2}{n^2}}{1+\frac{1}{n^2}}\right)$$

$\triangleleft$ As $n \to \infty,\ \frac{3}{n} \to 0,\ \frac{2}{n^2} \to 0,\ \frac{1}{n^2} \to 0$

$$=-4$$

$\therefore$ the sequence a_n converges to -4.

(c)

$u_r = 3^{-r}$

$u_r = \frac{1}{3^r}$

As $r \to \infty,\ 3^r \to \infty$

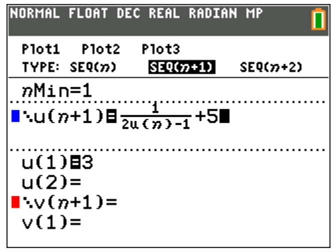
$\therefore\ \frac{1}{3^r} \to 0$

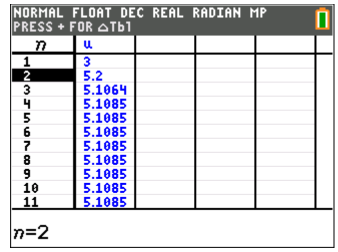
$\therefore$ the sequence u_r converges to 0.

Follow up 24

Solution

Reading in the table, the sequence decreases and converges to 5.108.





Follow up 25

Solution

Given $\quad S_n = 1 + \dfrac{n+1}{n+2}$

$$= 1 + \frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n} + \frac{2}{n}}$$

$$= 1 + \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}$$

As $n \to \infty$,

i.e. $\quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \dfrac{1 + \frac{1}{n}}{1 + \frac{2}{n}}\right)$

$$= 1 + 1$$

$$= 2$$

$S_\infty = 2$

Follow up 26

Solution

(a) Common ratio $= \dfrac{\text{second term}}{\text{first term}} = \dfrac{4}{5}$

(b) $S_\infty = \dfrac{a}{1-r}$

$$= \frac{5}{1 - \left(\frac{4}{5}\right)}$$

$$= 25$$

Follow up 27

Solution

Note that the series $2+x+\frac{x^2}{2}+\frac{x^3}{4}+$ is GP

Given that the series converges, therefore the sum to infinity exists.

The common ratio must be $|r|<1$.

i.e. $\left|\frac{x}{2}\right|<1$

$-2<x<2$

$$S_\infty=\frac{2}{1-\left(\frac{x}{2}\right)}$$

$$=\frac{4}{2-x}$$

Follow up 28

Solution

(a)

Given the first 2 terms of a geometric progression are x and y,

so the first term, $a = x$ and common ratio, $r = \frac{y}{x}$.

It is also given that sum of the first n terms is equal to the sum to infinity of the remaining terms.

i.e. $S_n = S_\infty - S_n$

$$2S_n = S_\infty$$

$$2\frac{a\left[1-\left(\frac{y}{x}\right)^n\right]}{1-\frac{y}{x}} = \frac{a}{1-\frac{y}{x}}$$

◁ Recall formula: $S_n = \frac{a(1-r^n)}{1-r}$ and $S_\infty = \frac{a}{1-r}$

$$2\left[1-\left(\frac{y}{x}\right)^n\right] = 1$$

$$2-2\left(\frac{y^n}{x^n}\right) = 1$$

$$2\left(\frac{y^n}{x^n}\right) = 1$$

$$2y^n = x^n \quad \text{(Proven)}$$

(b)

If $x = 1$ and $y = -\frac{1}{3}$, then GP : $a = 1, r = -\frac{1}{3}$.

$$S_n = \frac{1-\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right]$$

(c)

$$\left|S_n - S_\infty\right| < 0.001$$

$$\left|\frac{3}{4} - \frac{3}{4}\left[1 - \left(-\frac{1}{3}\right)^n\right]\right| < 0.001$$

$$\frac{3}{4}\left|1 - \left(-\frac{1}{3}\right)^n\right| < 0.001$$

$$\frac{3}{4}\left|\left(-\frac{1}{3}\right)^n\right| < 0.001$$

$$\left(\frac{1}{3}\right)^n < \frac{4}{3}(0.001)$$

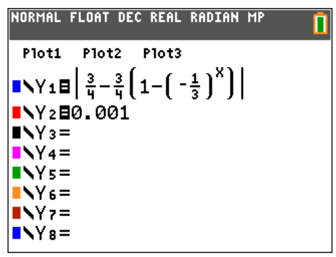
$$n \ln\left(\frac{1}{3}\right) < \ln\left[\frac{4}{3}(0.001)\right]$$

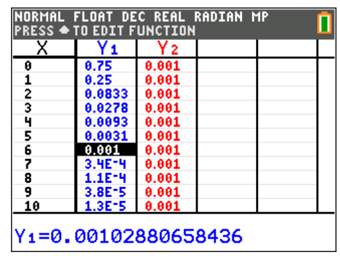
$$n > \frac{\ln \frac{4}{3}(0.001)}{\ln \frac{1}{3}}$$

$$n > 6.03$$

Alternatively, we can use GC to obtain the answer.

$$\left|\frac{3}{4} - \frac{3}{4}\left[1 - \left(-\frac{1}{3}\right)^n\right]\right| < 0.001$$





Follow up 29

Solution

(a) From the given information: $a = 2000$ and $r = 0.99$

Maximum number of watermelons which can be harvested from this farm in the long run

$= \dfrac{2\,000}{1-0.99}$ ◁ use sum of infinity formula

$= 200\,000$

(b) Let T_n be the month where the farmer harvests less than 500 watermelons.

$$T_n \le 500$$
$$2\,000(0.99)^{n-1} \le 500$$
$$(n-1)\ln 0.99 \ge \ln\left(\frac{500}{2\,000}\right) \quad \triangleleft \text{ Take ln both sides}$$
$$n-1 \ge \frac{\ln\left(\frac{500}{2\,000}\right)}{\ln 0.99}$$
$$n > 138.9$$

Hence, total number of months of havesting $= 139$

The last harvest month is on July 2019.

Follow up solutions C4 (1-6)

Follow up 1

Solution

(a) $\sum_{r=1}^{9} r^2 = 1+4+9+16+25+36+49+64+81$

(b) $\sum_{k=1}^{6} (11-8k) = 3-5-13-21-29-37$

Follow up 2

Solution

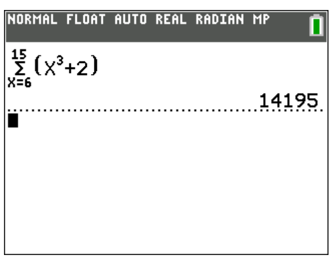
(a) $\sum_{r=0}^{\infty} 2^{-r} = 2^0 + 2^{-1} + 2^{-2} + \ldots$

(b) $\sum_{r=1}^{n} (-1)^{r+1}\left(\frac{x^r}{2r+1}\right) = \frac{x}{3} - \frac{x^2}{5} + \frac{x^3}{7} - \frac{x^4}{9} + \ldots + (-1)^{n+1}\left(\frac{x^n}{2n+1}\right)$

Follow up 3

Solution

$\sum_{r=6}^{15} (r^3+2) = 14195$



Follow up 4

Solution

(a) $4+8+16+32+64+128=\sum_{r=1}^{6} 2^{1+r}$

(b) $1\times 2+4\times 3+7\times 4+10\times 5+\ldots\text{to 20 terms}=\sum_{r=1}^{20}\left[(3r-2)(r+1)\right]$

Follow up 5

Solution

(a) $-\frac{1}{(4)(5)}+\frac{1}{(5)(6)}-\frac{1}{(6)(7)}+\ldots\ldots+\frac{1}{(2n+1)(2n+2)}=\sum_{r=3}^{2n}\frac{(-1)^r}{(r+1)(r+2)}$

(b) $1-x+x^2-x^3+\ldots=\sum_{r=1}^{\infty}(-1)^{1+r}x^{r-1}$

Follow up 6

Solution

(a) $\sum_{r=1}^{n}\left[r(3r-1)\right]$

$=\sum_{r=1}^{n}3r^2-\sum_{r=1}^{n}r$ ◁ use $\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{2}$ and $\sum_{r=1}^{n}r=\frac{n(n+1)}{2}$

$=3\left(\frac{n}{6}\right)(n+1)(2n+1)-\frac{n}{2}(n+1)$ ◁ take out common factor, $\frac{n}{2}(n+1)$

$=\frac{n}{2}(n+1)\left[(2n+1)+1\right]$

$=\frac{n}{2}(n+1)(2n+1-1)$

$=n^2(n+1)$

(b) $\sum_{r=1}^{2n}(3^r-10)$

$=\sum_{r=1}^{2n}3^r-\sum_{r=1}^{2n}10$ ◁ use $\sum_{r=1}^{n}a^r=\frac{a(r^n-1)}{r-1}$ and $\sum_{r=1}^{n}a=an$

$=\frac{3(3^{2n}-1)}{3-1}-(2n)10$

$=\frac{3(3^{2n}-1)}{2}-20n$

(c) $\sum_{k=1}^{30}(a^{2k}+ka)$

$=\sum_{k=1}^{30}(a^{2k})+a\sum_{k=1}^{30}k$

$=\frac{a^2\left(1-a^{2(30)}\right)}{1-a^2}+a\left(\frac{30}{2}\right)(30+1)$

$=\frac{a^2\left(1-a^{60}\right)}{1-a^2}+465a$

Follow up 7

Solution

(a) $\sum_{r=6}^{15}(r^3+2)$

$$=\sum_{r=6}^{15}r^3+\sum_{r=6}^{15}2$$

$$=\sum_{r=1}^{15}r^3-\sum_{r=1}^{5}r^3+(15-6+1)2$$

$$=\frac{1}{4}(15^2)(15+1)^2-\frac{1}{4}(5^2)(5+1)^2+20$$

$$=14195$$

(b) $\sum_{r=n}^{2n}(r+3n-n^r)$

$$=\sum_{r=n}^{2n}(r+3n)-\sum_{r=n}^{2n}(n^r)$$

$$=\frac{n+1}{2}(4n+5n)-\frac{n^n(n^{n+1}-1)}{n-1}$$

$$=\frac{9n(n+1)}{2}-\frac{n^n(n^{n+1}-1)}{n-1}$$

Follow up 8

Solution

(b) $\sum_{r=-2}^{n} \log_a(2a^r)$

$= \log_a(2a^{-2}) + \log_a(2a^{-1}) + \ldots + \log_a(2a^n)$

$= \log_a(2a^{-2} \times 2a^{-1} \times 2a^0 \times \ldots \times 2a^n)$

$= \log_a(2^{n-(-2)+1} a^{-2-1+0+1+2\ldots+n})$

$= \log_a\left(2^{n+3} a^{\frac{n+3}{2}(-2+n)}\right)$

Alternative Method

$\sum_{r=-2}^{n} (\log_a 2 + r)$

$= \sum_{r=-2}^{n} \log_a 2 + \sum_{r=-2}^{n} r$

$= (n+3)\log_a 2 + \frac{n+3}{2}(-2+n)$

$= \frac{1}{2}(n+3)(2\log_a(2) - 2 + n)$

Follow up 9

Solution

(a) $1^2+2^2+3^2+\ldots\ldots+(2n)^2$

$=\sum_{r=1}^{2n} r^2$

$=\frac{n}{6}(2n+1)(4n+1)$

(b) $2^2+4^2+6^2+\ldots\ldots+(2n)^2$

$=\sum_{r=1}^{n}(2r)^2$

$=4\sum_{r=1}^{n} r^2$

$=4\left[\frac{n}{6}(n+1)(2n+1)\right]$

$=\frac{2n}{3}(n+1)(2n+1)$

$1^2+3^2+5^2+\ldots+(2n-1)^2$

$=(1^2+2^2+3^2+\ldots+(2n)^2)-(2^3+4^2+6^2+\ldots+(2n)^2)$

$=\frac{n}{3}(2n+1)(4n+1)-\frac{2n}{3}(n+1)(2n+1)$ ⊲ obtained the result in **(a)** and **(b)**

$=\frac{n(2n+1)}{3}(4n+1-2n-2)=\frac{n}{3}(4n^2-1)$

Follow up 10

Solution

$$\sum_{n=1}^{N}(2^{n-2}+3n)$$

$$=\sum_{n=1}^{N}2^{n-2}+3\sum_{n=1}^{N}n$$

$$=(2^{-1}+2^{0}+\ldots+2^{N-2})+3(1+2+\ldots+N)$$

$$=\frac{2^{-1}(2^{N}-1)}{2-1}+3\left(\frac{N}{2}\right)(N+1)$$

$$=\frac{1}{2}[2^{N}-1+3N(N+1)]$$

Follow up solutions C7 (1-10)

Follow up 1

Solution

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{(x+\delta x) - x}$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{1}{(x+\delta x)^2} - \dfrac{1}{x^2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{x^2 - (x+\delta x)^2}{(x+\delta x)^2 x^2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2 - (x^2 + 2x\delta x + (\delta x)^2)}{\delta x(x+\delta x)^2 x^2}$$

$$= \lim_{\delta x \to 0} \frac{-2x\delta x - (\delta x)^2}{\delta x(x+\delta x)^2 x^2}$$

$$= \lim_{\delta x \to 0} \frac{-2x - (\delta x)}{(x+\delta x)^2 x^2}$$

$$= \frac{-2x}{(x)^2 x^2}$$

$$= -\frac{2}{x^3}$$

Follow up 2

Solution

$$y = x^{\frac{1}{3}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}x^{-\frac{2}{3}}$$

Follow up 3

Solution

$$y = ax^3$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2$$

Follow up 4

Solution

$$y = x^2 - 2x^3 + \frac{2}{3}x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6x^2 + \frac{2}{3}$$

Follow up 5

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 - 5x^{\frac{1}{3}}\right)^2$$
$$= 2\left(2x^3 - 5x^{\frac{1}{3}}\right)\left(6x^2 - \frac{5}{3}x^{-\frac{2}{3}}\right)$$
$$= 2\left(2x^3 - 5x^{\frac{1}{3}}\right)\left(6x^2 - \frac{5}{3x^{\frac{2}{3}}}\right)$$

Follow up 6

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{5x-2}}\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left((5x-2)^{-\frac{1}{2}}\right)$$

$$=\left(-\frac{1}{2}\right)(5x-2)^{-\frac{3}{2}}(5)$$

$$=-\frac{5}{2(5x-2)^{\frac{3}{2}}}$$

Learning Point:

It is not correct to write your working as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}=\left(\frac{1}{\sqrt{5x-2}}\right)$$

Follow up 7

Solution

$$y = x^3(x+1)^2$$

$$\frac{dy}{dx} = x^3\frac{d}{dx}(x+1)^2 + (x+1)^2\frac{d}{dx}(x^3) \qquad \triangleleft \text{ use product rule}$$

$$= x^3 \times 2(x+1) + (x+1)^2 \times 3x^2$$

$$= 2x^3(x+1) + 3x^2(x+1)^2$$

$$= x^2(x+1)[2x+3(x+1)]$$

$$= x^2(x+1)(2x+3x+3)$$

$$= x^2(x+1)(5x+3)$$

Hence $\frac{dy}{dx} = x^2(x+1)(5x+3)$

Follow up 8

Solution

$$y = \frac{x}{2x+5}$$

$$y = x(2x+5)^{-1}$$

$$\frac{dy}{dx} = -x(2x+5)^{-2}(2) + (2x+5)^{-1}(1)$$

$$= -\frac{2x}{(2x+5)^2} + \frac{1}{(2x+5)}$$

$$= \frac{-2x+(2x+5)}{(2x+5)^2}$$

$$= \frac{5}{(2x+5)^2}$$

Follow up 9

Solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{(4t+1)^{\frac{1}{2}}}\right) \qquad \text{< use quotient rule}$$

$$=\frac{(4t+1)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}(t)-t\frac{\mathrm{d}}{\mathrm{d}t}(4t+1)^{\frac{1}{2}}}{(\sqrt{4t+1})^2}$$

$$=\frac{(4t+1)^{\frac{1}{2}}(1)-t\left(\frac{1}{2}\right)(4t+1)^{-\frac{1}{2}}(4)}{(\sqrt{4t+1})^2}$$

$$=\frac{(4t+1)^{\frac{1}{2}}-2t(4t+1)^{-\frac{1}{2}}}{(4t+1)}$$

$$=\frac{(4t+1)^{-\frac{1}{2}}\left[(4t+1)-2t\right]}{(4t+1)}$$

$$=\frac{(2t+1)}{(4t+1)^{\frac{1}{2}}(4t+1)}$$

$$=\frac{2t+1}{(2x+1)^{\frac{3}{2}}}$$

Follow up 10

Solution

(a) $y = 3\tan(1-2x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sec^2(1-2x)\frac{\mathrm{d}}{\mathrm{d}x}(1-2x)$$

$$= 3\sec^2(1-2x)\times(-2)$$

$$= -6\sec^2(1-2x)$$

(b) $y = \operatorname{cosec}\left(3x-\frac{\pi}{5}\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec}\left(3x-\frac{\pi}{5}\right)\cot\left(3x-\frac{\pi}{5}\right)\times\frac{\mathrm{d}}{\mathrm{d}x}\left(3x-\frac{\pi}{5}\right)$$

$$= -\operatorname{cosec}\left(3x-\frac{\pi}{5}\right)\cot\left(3x-\frac{\pi}{5}\right)\times(3)$$

$$= -3\operatorname{cosec}\left(3x-\frac{\pi}{5}\right)\cot\left(3x-\frac{\pi}{5}\right)$$

Follow up solutions C7 (11-20)

Follow up 11

Solution

(a) Let $y = 2\sec^3(1-x)$

$$y = 2[\sec(1-x)]^3$$

$$\frac{dy}{dx} = 6[\sec(1-x)]^2 \tan(1-x)\sec(1-x)\frac{d}{dx}(1-x)$$

$$= 6\sec^3(1-x)\tan(1-x)\times(-1)$$

$$= -6\sec^3(1-x)\tan(1-x)$$

Learning Point:

Given $\left[\sin f(x)\right]^n$, we can express

$\left[\sin f(x)\right]^n = \sin^n f(x)$

For example, $\left[\sin f(x)\right]^2 = \sin^2 f(x)$

The above result applies for all the trigonometric functions.

(b) Let $y = \cos(x^4)$

$$\frac{dy}{dx} = -\sin(x^4)\frac{d}{dx}(x^4)$$

$$= \sin(x^4)\times(4x^3)$$

$$= 4x^3\sin(x^4)$$

Learning Point:

Given $\sin(f(x))^n$, it is not correct to say

$\sin(f(x))^n \neq \sin^n f(x)$ or $\sin(f(x))^n \neq [\sin(f(x)]^n$

For example, $\sin x^2 \neq \sin^2 x$.

The above result applies for all the trigonometric functions.

Follow up 12

Solution

(a) Let $y = \operatorname{cosec}\left(\frac{\pi}{4} - 2x^3\right)$

$$\frac{dy}{dx} = -\operatorname{cosec}\left(\frac{\pi}{4} - 2x^3\right)\cot\left(\frac{\pi}{4} - 2x^3\right)\frac{d}{dx}\left(\frac{\pi}{4} - 2x^3\right)$$

$$= -\operatorname{cosec}\left(\frac{\pi}{4} - 2x^3\right)\cot\left(\frac{\pi}{4} - 2x^3\right)\times(-6x^2)$$

$$= 6x^2 \operatorname{cosec}\left(\frac{\pi}{4} - 2x^3\right)\cot\left(\frac{\pi}{4} - 2x^3\right)$$

(b) Let $y = \frac{1}{\sqrt{\cot 3x}}$

$$= (\cot 3x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}(\cot 3x)^{-\frac{3}{2}}(-\operatorname{cosec}^2 3x)\frac{d}{dx}(3x)$$

$$= \frac{1}{2}(\cot 3x)^{-\frac{3}{2}}(\operatorname{cosec}^2 3x)\times 3$$

$$= \frac{3x \operatorname{cosec}^2 3x}{2\sqrt{(\cot 3x)^3}}$$

Follow up 13

Solution

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^2\sqrt{1-2x}$$

$$= 2\cos\sqrt{1-2x}\times(-\sin\sqrt{1-2x})\frac{\mathrm{d}}{\mathrm{d}x}(1-2x)^{\frac{1}{2}}$$

$$= -2\sin\sqrt{1-2x}\cos\sqrt{1-2x}\times\left(\frac{1}{2}(1-3x)^{-\frac{1}{2}}(-2)\right)$$

$$= \frac{2\sin\sqrt{1-2x}\cos\sqrt{1-2x}}{\sqrt{1-2x}}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec(2x+3))^3$$

$$= 3(\sec(2x+3))^3\sec(2x+3)\tan(2x+3)(2)$$

$$= 6\sec^3(2x+3)\tan(2x+3)$$

Follow up 14

Solution

(a) $y = \cos 2x \cot^3 x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x(3\cot^2 x)(-\operatorname{cosec}^2 x) + \cot^3 x(-2\sin 2x)$$

$$= -3\cot^2 x \cos 2x \operatorname{cosec}^2 x - 2\sin 2x \cot^3 x)$$

$$= -\cot^2 x(3\cos 2x \operatorname{cosec}^2 x + 2\sin 2x \cot x)$$

(b) $y = \dfrac{\sin x}{\sqrt{1-\sec x}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1-\sec x}(\cos x) - \sin x \frac{1}{2}(1-\sec x)^{-\frac{1}{2}}(-\sec x \tan x)}{1-\sec x}$$

$$= \frac{\frac{1}{\sqrt{1-\sec x}}\left((1-\sec x)\cos x + \frac{1}{2}\sin x(\sec x \tan x)\right)}{1-\sec x}$$

$$= \frac{\frac{1}{2}\left[2(1-\sec x)\cos x + \sin x(\sec x + \tan x)\right]}{(1-\sec x)^{\frac{3}{2}}}$$

$$= \frac{2\cos x - 2 + \sin x \sec x + \sin x \tan x}{2(1-\sec x)^{\frac{3}{2}}}$$

$$= \frac{2\cos x - 2 + \tan x + \sin x \tan x}{2(1-\sec x)^{\frac{3}{2}}}$$

Follow up 15

Solution

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos^{-1} 3\theta)$$
$$= -\frac{3}{\sqrt{1-(3\theta)^2}}$$
$$= -\frac{3}{\sqrt{1-9\theta^2}}$$

Follow up 16

Solution

(a) $y = -4\tan^{-1}(1-2x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4}{1+(-2x)^2} \times \frac{\mathrm{d}}{\mathrm{d}x}(1-2x)$$
$$= \frac{-4}{1+4x^2} \times (-2)$$
$$= \frac{8}{1+4x^2}$$

(b) $y = \sin^{-1}(\ln x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-(\ln x)^2}} \times \frac{\mathrm{d}}{\mathrm{d}x}(\ln x)$$
$$= \frac{1}{\sqrt{1-(\ln x)^2}} \times \left(\frac{1}{x}\right)$$
$$= \frac{1}{x\sqrt{1-(\ln x)^2}}$$

Follow up 17

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{1-x})$$
$$=-\mathrm{e}^{1-x}$$

Follow up 18

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\tan^{-1}2x})$$
$$=\frac{2}{1+(2x)^2}\times\mathrm{e}^{\tan^{-1}2x}$$
$$=\frac{2\mathrm{e}^{\tan^{-1}2x}}{1+4x^2}$$

Follow up 19

Solution

$$\begin{aligned} y &= 4^{\cos x} \\ &= (\ln 4)(4^{\cos x})\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 4(4^{\cos x})(-\sin x) \\ &= -\ln 4\sin x(4^{\cos x}) \end{aligned}$$

Follow up 20

Solution

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\ln(1-2t)$

$= -\dfrac{2}{1-2t}$

(b) $\mathrm{f}\,'(x) = \dfrac{2\sec^2 2x}{\tan 2x}$

$= \dfrac{2}{\cos^2 2x}\left(\dfrac{\cos 2x}{\sin 2x}\right)$

$= \dfrac{2}{\cos 2x \sin 2x}$

$= \dfrac{4}{2\sin 2x \cos 2x}$

$= \dfrac{4}{\sin 4x}$

$= 4\,\mathrm{cosec} 2x$

Follow up 21

Solution

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x)^{-2}$

$= -2(\ln x)^{-3} \times \dfrac{1}{x}$

$= -\dfrac{2}{x(\ln x)^3}$

(b) $\mathrm{f}(x) = \ln(\ln x^2)$

$= \ln(2\ln x)$

$\mathrm{f}'(x) = \dfrac{\frac{2}{x}}{2\ln x}$

$= \dfrac{1}{x\ln x}$

Follow up 22

Solution

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{a^2-x^2}\right)$ $\quad \triangleleft \ln a^b = b\ln a$

$= \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2}\ln(a^2-x^2)\right)$

$= \dfrac{1}{2}\left[\dfrac{-2x}{a^2-x^2}\right]$

$= -\dfrac{x}{a^2-x^2}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\lg(\tan x+1)$

$= \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\ln 10}\lg(\tan x+1)\right)$

$= \dfrac{1}{\ln 10}\left(\dfrac{\sec^2 x}{\tan x+1}\right)$

Follow up 23

Solution

$$\begin{aligned} \mathrm{f}(x) &= \ln \mathrm{e}^x + \ln(\sin x) \quad \lhd \ln ab = \ln a + \ln b \\ &= x \ln \mathrm{e} + \ln(\sin x) \\ &= x + \ln(\sin x) \end{aligned}$$

$$\begin{aligned} \mathrm{f}'(x) &= 1 + \frac{\cos x}{\sin x} \\ &= 1 + \cot x \end{aligned}$$

Follow up 24

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln \sqrt{\frac{x^2+1}{3x^2+a}} \right) \quad \lhd \ln a^b = b \ln a$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+1) - \ln(3x^2+a) \right] \quad \lhd \ln ab = \ln a + \ln b$$

$$= \frac{1}{2}\left[\frac{2x}{x^2+1} - \frac{6x}{3x^2+a} \right]$$

Follow up 25

Solution

$$\frac{d}{dx}(\sin y + 3y^2)$$
$$= \cos y\frac{dy}{dx} + 6y\frac{dy}{dx}$$

Follow up 26

Solution

$(\ln y)^2 - 3xy + x^2 - 4 = 0$

Differentiate both sides with respect to x:

$$2\ln y\left(\frac{1}{y}\right)\frac{dy}{dx} - \left(3x\frac{dy}{dx} + 3y\right) + 2x = 0$$
$$2\ln y\left(\frac{1}{y}\right)\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y + 2x = 0$$
$$\left(\frac{2\ln y}{y} - 3x\right)\frac{dy}{dx} = 3y - 2x$$
$$\therefore \quad \frac{dy}{dx} = \frac{(3y-2x)y}{2\ln y - 3xy}$$

Follow up 27

Solution

Method 1:

Let $y = x^{\sec 2x}$

Take ln on both sides: $\ln y = (\sec 2x)(\ln x)$ $\lhd$ apply product rule on RHS

Differentiate both sides with respect to x:

$$\frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = (\sec 2x)\left(\frac{1}{x}\right) + (2\sec 2x \cdot \tan 2x)(\ln x) \quad \lhd \text{ factor out } \sec 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\sec 2x\left[\frac{1}{x} + 2(\tan 2x)(\ln x)\right] \quad \lhd \text{ replace } y \text{ by } x^{\sec 2x}$$

$$= x^{\sec 2x}(\sec 2x)\left[\frac{1}{x} + 2(\tan 2x)(\ln x)\right]$$

Method 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\sec 2x})$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\ln(x^{\sec 2x})}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{(\sec 2x)(\ln x)}\right)$$

$$= \mathrm{e}^{(\sec 2x)(\ln x)} \times \frac{\mathrm{d}}{\mathrm{d}x}((\sec 2x)(\ln x))$$

$$= \mathrm{e}^{(\sec 2x)(\ln x)} \underbrace{(\sec 2x)\left(\frac{1}{x}\right) + (2\sec 2x \cdot \tan 2x)(\ln x)}_{\text{Product Rule}}$$

$$= x^{\sec 2x}(\sec 2x)\left[\frac{1}{x} + 2(\tan 2x)(\ln x)\right]$$

Follow up 28

Solution

Given $\qquad y = \sqrt{\mathrm{e}^x \cos x}$

$\qquad\qquad y^2 = \mathrm{e}^x \cos x \qquad \lhd$ square both sides

Differentiate both sides with respect to x,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - \mathrm{e}^x \sin x$$

Differentiate both sides with respect to x,

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - (\mathrm{e}^x \sin x + \mathrm{e}^x \cos x)$$ $\lhd$ replace $\mathrm{e}^x \sin x$ by $2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2$ and replace $\mathrm{e}^x \cos x$ by y^2

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} + \left(2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2\right) - y^2$$

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4y\frac{\mathrm{d}y}{\mathrm{d}x} - 2y^2$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 \quad \text{(Shown)}$$

Follow up 29

Solution

Given $\quad x = (2t+1)^2$

Differentiate with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(2t+1)\times 2$$
$$= 4(2t+1)$$

Also given $y = 2t^3$

Differentiate with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6t^2$$

Using the Chain Rule, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$

$$= \frac{6t^2}{4(2t+1)}$$
$$= \frac{3t^2}{2(2t+1)}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3t^2}{2(2t+1)}\right)^*}{4(2t+1)} \quad \triangleleft \text{ see side working}$$
$$= \frac{3t(t+1)}{2(2t+1)^2}\times\frac{1}{4(2t+1)}$$
$$= \frac{3t(t+1)}{8(2t+1)^3}$$

$$* \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3^2}{8t+4}\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3t^2}{8t+4}\right]$$
$$= \left(\frac{(8t+4)\frac{\mathrm{d}}{\mathrm{d}x}[3t^2] - 3x^2\frac{\mathrm{d}}{\mathrm{d}x}[8t+4]}{(8t+4)^2}\right)$$
$$= \frac{6t(8t+4)-24t^2}{(8t+4)^2}$$
$$= \frac{48t^2+24t-24t^2}{4^2(2t+1)^2}$$
$$= \frac{24t+24t^2}{4^2(2t+1)^2}$$
$$= \frac{3t(t+1)}{2(2t+1)^2}$$

Follow up 30

Solution

Given $x = t - \sin t$

Differentiate with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \cos t$$

Also given $y = 1 - \cos t$

Differentiate with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sin t$$

Using the Chain Rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$

$$= \sin t \times \frac{1}{1 - \cos t}$$

$$= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{1 - \left(1 - 2\sin^2\frac{t}{2}\right)} \quad \triangleleft \text{ use double angle}$$

$$= \cot\frac{t}{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\cot\frac{t}{2}\right)}{2\sin^2\frac{t}{2}}$$

$$= -\frac{1}{2}\operatorname{cosec}^2\left(\frac{t}{2}\right) \times \frac{1}{2\sin^2\frac{t}{2}}$$

$$= -\frac{1}{4}\operatorname{cosec}^4\left(\frac{t}{2}\right)$$

Follow up solutions C8 (1-10)

Follow up 1

Solution

$$y = x(2x-1)(x+3)$$
$$y = 2x^3 + 5x^2 - 3x$$
$$\frac{dy}{dx} = 6x^2 + 10x - 3$$

When $x = 1, \frac{dy}{dx} = 6(1)^2 + 10(1) - 3$
$$= 13$$

Gradient of the curve at (1, 4) is 13.

Follow up 2

Solution

$$y = 2x^2 + 8x^{-2} - 9$$
$$\frac{dy}{dx} = 4x - 16x^{-3}$$

When $x = 2$, $y = 2(2)^2 + 8(2)^{-2} - 9$
$$= 1$$

and $\frac{dy}{dx} = 4(2) - 16(2)^{-3}$
$$= 6$$

Equation of tangent passes through the point (2, 1) with gradient = 6 is given by

$$y - 1 = 6(x-2)$$
$$y = 6x - 11$$

Gradient of normal $= -\frac{1}{6}$ $\quad \triangleleft\ m_1 m_2 = -1$

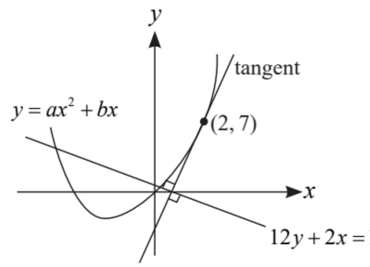
Equation of normal passes through the point (2, 1) with gradient $= -\frac{1}{6}$ is given by

$$y - 1 = -\frac{1}{6}(x-2)$$
$$-6y + 6 = x - 2$$
$$x + 6y = 8$$

Follow up 3

Solution

From the diagram, we note that the point (2, 7) lies on the curve.



Substituting (2, 7) into $y = ax^2 + bx$.

$$7 = a(2)^2 + b(2)$$

$$7 = 4a + 2b \text{ .... (1)}$$

Gradient of the line $12y + 2x = 1$ is $-\frac{1}{6}$.

The line $12y + 2x = 1$ is $\perp$ to the tangent.

$\therefore$ gradient of the tangent at (2, 7) = 6

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b \qquad \triangleleft \text{ substituting } \frac{\mathrm{d}y}{\mathrm{d}x} = 6 \text{ and } x = 2$$

$$6 = 2a(2) + b$$

$$6 = 4a + b \text{ .... (2)}$$

Taking $(1) - (2)$

$$b = 1$$

Substituting (2, 7) and $b = 1$ into (1).

$$7 = 4a + 2$$

$$5 = 4a$$

$$a = \frac{5}{4}$$

$\therefore\ a = \frac{5}{4}$ and $b = 1$

Follow up 4

Solution

(a) $x = 1 + \cos\theta$ $\qquad\qquad$ $y = \sin\theta$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta$ $\qquad\qquad$ $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta$

Using chain rule

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}\\ &= -\frac{\cos\theta}{\sin\theta}\\ &= -\cot\theta\end{aligned}$$

Substitute $x = 1$ into $x = 1 + \cos\theta$ to obtain the parameter θ.

$$\begin{aligned}1 &= 1 + \cos\theta\\ 0 &= \cos\theta\\ \theta &= \frac{\pi}{2}\end{aligned}$$

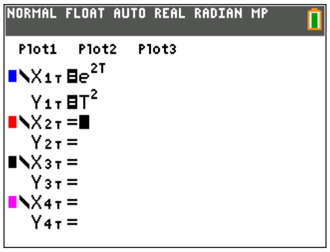
Substitute $\theta = \dfrac{\pi}{2}$ into $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\cot\theta$.

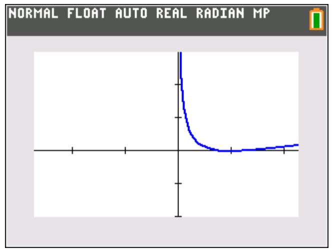
$\therefore \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

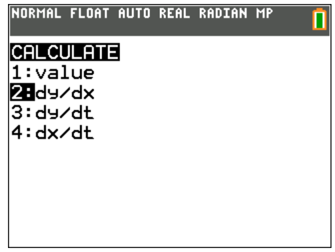
Follow up 5

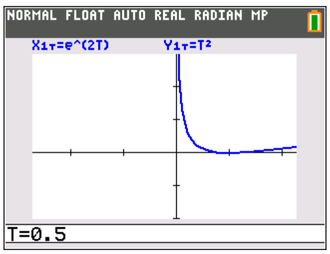
Solution

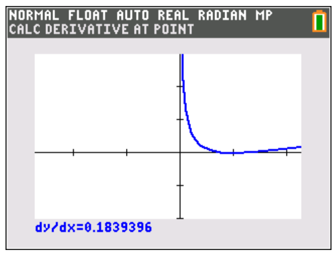
Using GC, we can obtain gradient at the point on the curve at the given parameter.











From GC, the gradient at the point on the curve where $t = \dfrac{1}{2}$ is 0.184.

Alternative Method (analytical Method)

Solution

Given $\quad x = e^{2t}$ .... (1)

and $\quad y = t^2$ .... (2)

Differentiate (1) with respect to t

$$\frac{dx}{dt} = 2e^{2t}$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = 2t$$

Using the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dx}{dt} \\ &= 2t \times \frac{1}{2e^{2t}} \\ &= \frac{t}{e^{2t}} \end{aligned}$$

When $t = \dfrac{1}{2}$

$$\frac{dy}{dx} = \frac{1}{2e}$$

Using chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= 2t \times \frac{1}{2\mathrm{e}^{2t}}$$

$$= \frac{t}{\mathrm{e}^{2t}}$$

When $t = 0.5$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.184$

The gradient at the point on the curve where $t = \frac{1}{2}$ is 0.184.

Follow up 6

Solution

(a) Given $\quad x = a\cos^3\theta$ .... (1)

and $\quad y = a\sin^3\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3a\cos^2\theta(-\sin\theta)$$
$$= -3a\cos^2\theta\sin\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta(\cos\theta)$$
$$= 3a\sin^2\theta\cos\theta$$

Using chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}x}$$
$$= \frac{3a\sin^2\theta(\cos\theta)}{-3a\cos^2\theta\sin\theta}$$
$$= -\frac{\sin\theta}{\cos\theta}$$
$$= -\tan\theta \qquad \text{(Shown)}$$

(b) Given point $(a\cos^3 p, a\sin^3 p)$

Substitute $x = a\cos^3 p$ into $x = a\cos^3\theta$ to obtain the parameter θ.

$\therefore\ \theta = p$

When $\theta = p$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\tan p$

$\therefore$ gradient of tangent at $\theta = p$ is $-\tan p$.

Equation of tangent to the curve at $(a\cos^3 p, a\sin^3 p)$:

$$y - a\sin^3 p = -\tan p(x - a\cos^3 p) \qquad \triangleleft \text{ use } y - y_1 = m(x - x_1)$$
$$y = -(\tan p)x + a\sin^3 p + a\tan p\cos^3 p$$

Gradient of the normal at $\theta = p$ is $\dfrac{1}{\tan p}$. $\qquad \triangleleft\ m_1 m_2 = -1$

Equation of normal to the curve at $(a\cos^3 p, a\sin^3 p)$:

$$y - a\sin^3 p = \frac{1}{\tan p}(x - a\cos^3 p)$$
$$y = \frac{1}{\tan p}x + a\sin^3 p - a\frac{\cos^3 p}{\tan p}$$

Follow up 7

Solution

Given $x = t^2 + 1$ .... (1)

and $y = t^3 - t$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 1$$

Using chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= (3t^2 - 1) \times \frac{1}{2t}$$

$$= \frac{3t^2 - 1}{2t}$$

Given that the curve has a tangent which is parallel to the line $y = x + 5$, i.e. the gradient of this line is 1.

$$\therefore \quad \frac{3t^2 - 1}{2t} = 1$$

$$(3t + 1)(t - 1) = 0$$

$$\therefore \quad t = -\frac{1}{3} \text{ (NA)} \quad \text{or } 1$$

Putting $t = 1$ into (1) and (2) gives

$x = (1)^2 + 1 = 2$ and $y = (1)^3 - 1 = 0$.

$\therefore$ the coordinates of the point when $t = 1$ are (2, 0)

Substitute $t = 1$ into the parametric equations:

$$x = (1)^2 + 1 \qquad\qquad y = (1)^3 - (1)$$

$$= 2 \qquad\qquad\qquad = 0$$

$\therefore$ the coordinates is (2, 0).

Equation of the tangent at (2, 0) is

$$y - 0 = 1(x - 2)$$

$$y = x - 2$$

$\therefore$ $y = x - 2$

Follow up 8

Solution

(a) Given $x = 2t$ .... (1)

$$y = \frac{1}{t^2} \text{ .... (2)}$$

Substituting (1) into (2).

$$y = \frac{1}{\left(\frac{x}{2}\right)^2}$$

$$= \frac{4}{x^2}$$

$\therefore$ the cartesian equation the curve is $y = \dfrac{4}{x^2}$. ....(3)

(b) Differentiate with (3) respect to x

$$\frac{dy}{dx} = -\frac{8}{x^3}.$$

At $x = 2t$,

$$\frac{dy}{dx} = -\frac{8}{(2t)^3}$$

$$= -\frac{1}{t^3}$$

Equation of tangent at $\left(2t, \dfrac{1}{t^2}\right)$:

$$y - \frac{1}{t^2} = -\frac{1}{t^3}(x - 2t)$$

$$y - \frac{1}{t^2} = -\frac{1}{t^3}x - \frac{2}{t^2}$$

$$\therefore\ y = \frac{3}{t^2} - \frac{x}{t^3}$$

Equation of normal at $\left(2t, \dfrac{1}{t^2}\right)$:

$$y - \frac{1}{t^2} = t^3(x - 2t)$$

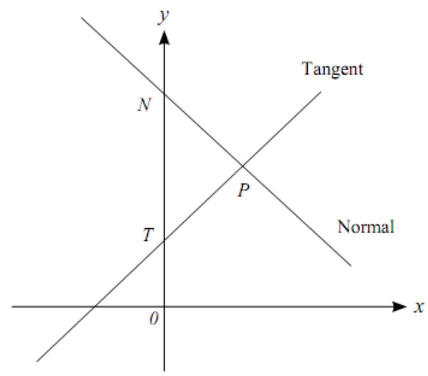
$$y - \frac{1}{t^2} = t^3 x - 2t^4$$

$$\therefore\ y = t^3 x + \frac{1}{t^2} - 2t^4$$

(c) At the point $P\left(4, \frac{1}{4}\right)$, the parameter of t is 2.

Substituting $t = 2$ into the equation of tangent $y = \frac{3}{t^2} - \frac{x}{t^3}$.

$$y = \frac{3}{2^2} - \frac{x}{2^3}$$
$$y = \frac{3}{4} - \frac{1}{8}x$$



When the tangent cuts the y-axis, i.e $x = 0$.

Substituting $x = 0$ into $y = \frac{3}{4} - \frac{1}{8}x$.

$$y = \frac{3}{4} - \frac{1}{8}(0)$$
$$y = \frac{3}{4}$$

$\therefore\ T\left(0, \frac{3}{4}\right)$

Substituting $t = 2$ into the equation of normal $y = t^3 x + \frac{1}{t^2} - 2t^4$.

$$y = (2)^3 x + \frac{1}{2^2} - 2(2)^4$$
$$y = 8x - \frac{127}{4}$$

When the normal cuts the y-axis, i.e $x = 0$.

Substituting $x = 0$ into $y = 8x - \frac{127}{4}$.

$$y = 8(0) - \frac{127}{4}$$
$$y = -\frac{127}{4}$$

$\therefore\ N\left(0, -\frac{127}{4}\right)$

Area of ΔPTN

$= \frac{1}{2}$(Perpendicular distance from P to the y-axis)(TN)

$= \frac{1}{2}(4)\left(\frac{3}{4} + \frac{127}{4}\right)$

$= 65$ Units^2 (Shown)

Follow up 9

Solution

(a)

Given $x = at^2$ .... (1) $y = 2at$ .... (2)

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= 2a \times \frac{1}{2at}$$
$$= \frac{1}{t}$$

Since gradient of tangent $= \frac{1}{t}$. $\therefore$ the gradient of normal $= -t$.

Equation of normal to the curve at any point with parameter t

$$y - 2at = -t(x - at^2)$$
$$y - 2at = -tx + at^3$$

$\therefore$ $y + tx = 2at + at^3$ (shown) .... (3)

(b) Substituting $t = -1$ into (3).

$$y - x = -2a - a$$

$\therefore$ $y = x - 3a$ .... (4)

The normal cuts the curve again at R.

Substituting (1) and (2) into (4)

$$2at = at^2 - 3a$$
$$2t = t^2 - 3$$

$\therefore$ $t^2 - 2t - 3 = 0$

$t = 3$ or -1 (Rejected)

Substituting $t = 3$ into (1):

$$x = a(3)^2$$
$$= 9a$$

Substituting $t = 3$ into (2):

$$y = 2a(3)$$
$$y = 6a$$

The coordinates of R are $(9a, 6a)$.

(c) Given that the line and curve intersect,

Substituting (1) and (2) into $x - yp + a = 0$

So, $at^2 - p(2at) + a = 0$

$at^2 - 2pat + a = 0$

Given that they cut at 2 distinct points, Discriminant > 0.

i.e. $(-2pa)^2 - 4a(a) > 0$

$4p^2a^2 - 4a^2 > 0$

$(p+1)(p-1) > 0$

$\therefore \ p > 1$ or $p < -1$

Follow up 10

Solution

Let $(x-1)^2-(y+4)^2=16$ .... (1)

Differentiating (1) wrt x,

$$2(x-1)-2(y+4)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$2(y+4)\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-1}{y+4} \quad \text{...............} \ (2)$$

Substituting the point $(3,\ 2)$ into (2).

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{3-1}{2+4}\\&=3\end{aligned}$$

At $(3,\ 2)$, gradient of the tangent is 3.

Follow up 11

Solution

(a) $x^2 - 3xy + 2y^2 = 6$ .... (1)

Differentiating (1) implicitly wrt x

$$2x - 3x\frac{dy}{dx} - 3y + 4y\frac{dy}{dx} = 0$$

$$(3x - 4y)\frac{dy}{dx} = 2x - 3y$$

$$\frac{dy}{dx} = \frac{2x - 3y}{3x - 4y}$$

Substituting the point $(2,3)$ into (1) gives

$$\frac{dy}{dx} = \frac{2(2) - 3(3)}{3(2) - 4(3)}$$

$$= \frac{5}{6}$$

Equation of tangent at $(2,3)$ is

$$y - 3 = \frac{5}{6}(x - 2)$$

$$6y = 5x + 8$$

(b) For tangent parallel to the x-axis, $\dfrac{dy}{dx} = 0.$

i.e $\quad \dfrac{2x - 3y}{3x - 4y} = 0$

$$2x - 3y = 0$$

$$y = \frac{2}{3}x$$

Substituting $y = \frac{2}{3}x$ into (1).

$$x^2 - 3x\left(\frac{2}{3}x\right) + 2\left(\frac{2}{3}x\right)^2 = 6$$

$$-9x^2 + 8x^2 = 54$$

$$x^2 = -54$$

Since $x^2 \geq 0, x \in \mathbb{R}$. Thus, there is no point on the curve at which the tangent is parallel to the x-axis. (Shown)

(c) Let $y = a$, where a is a constant .... (2)

Substituting (2) into (1)

$$x^2 - 3xa + 2a^2 = 6$$

$$\therefore \; x^2 - 3xa + (2a^2 - 6) = 0$$

Using the Discriminant, D

$= (-3a)^2 - 4(2a^2 - 6)$

$= a^2 + 24$

For any real values of a, $a^2 \geq 0$. $\therefore$ $a^2 + 24$ is always positive.

Since discriminant, $D > 0$, $\therefore$ every line parallel to the x-axis cuts the curve at two distinct points. (Shown)

B Follow up solutions C8 (12-21)

Follow up 12

Solution

$$\mathrm{f}(x)=\frac{x}{\mathrm{e}^x}$$

$$\mathrm{f}'(x)=\frac{\mathrm{e}^x(1)-x(\mathrm{e}^x)}{(\mathrm{e}^x)^2}$$

$$\mathrm{f}'(x)=\frac{1-x}{\mathrm{e}^x}$$

For $\mathrm{f}(x)$ is increasing, $\mathrm{f}'(x)\geq 0$.

$$\therefore\ \frac{1-x}{\mathrm{e}^x}\geq 0$$

e^x is always positive for all real values of x,

$$\therefore\ 1-x\geq 0$$

$$x-1\leq 0$$

$$x\leq 1$$

Range of values of x is $0<x\leq 1$.

Follow up 13

Solution

$$f(x)=\frac{2x}{(x+1)(x-2)}$$

$$f'(x)=\frac{(x+1)(x-2)(2)-(2x)(2x-1)}{\left[(x+1)(x-2)\right]^2}$$

$$=\frac{-2x^2-4}{\left[(x+1)(x-2)\right]^2}$$

$$=-\frac{2x^2+4}{\left[(x+1)(x-2)\right]^2}$$

Since $\left[(x+1)(x-2)\right]^2$ is always positive, for all real values of $x,\ x\neq -1,\ x\neq 2$.

$2x^2+4$ is always positive , for all real values of x.

So, $f'(x)=-\dfrac{2x^2+4}{\left[(x+1)(x-2)\right]^2}<0,$

i.e. $f'(x)$ is always negative.

$\therefore\ f(x)$ is strictly decreasing for all values of $x,\ x\neq -1,\ x\neq 2$.

Follow up 14

Solution

$$y = x^3 + x^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2x - 1$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x + 2$$

(a) For the curve concave upward, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$

i.e. $6x + 2 > 0$

$$x > -\frac{1}{3}$$

(b) For the curve concave downward, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$

$$6x + 2 < 0$$

$$x < -\frac{1}{3}$$

Follow up 15

Solution

(a) $y = \dfrac{x^3}{1+3x^4}$ .... (1)

$$\frac{dy}{dx} = \frac{(1+3x^4)3x^2 - x^3(12x^3)}{(1+3x^4)^2}$$

$$= \frac{3x^2 - 3x^6}{(1+3x^4)^2}$$

$$= \frac{3x^2(1-x^4)}{(1+3x^4)^2}$$

At stationary, $\dfrac{dy}{dx} = 0$.

$\therefore \quad 3x^2(1-x^4) = 0$

$x = 0$ or 1 or -1

Substituting $x = 0$ into (1). $\therefore \; y = 0$

Substituting $x = 1$ into (1). $\therefore \; y = \dfrac{1}{4}$

Substituting $x = -1$ into (1). $\therefore \; y = -\dfrac{1}{4}$

The coordinates of the stationary points are $(0, 0)$, $\left(1, \dfrac{1}{4}\right)$ and $\left(-1, -\dfrac{1}{4}\right)$.

(b) Using First Derivative Test

x	0^-	0	0^+
$\dfrac{dy}{dx}$	$+$	0	$+$
direction of tangent	/	—	/
shape of curve			

The stationary point $(0, 0)$ is a point of inflexion.

x	$(-1)^-$	1	$(-1)^+$
$\dfrac{dy}{dx}$	$-$	0	$+$
direction of tangent	\	—	/
shape of curve			

The stationary point $\left(-1, -\dfrac{1}{4}\right)$ is a minimum point.

x	1^-	1	1^+
$\frac{dy}{dx}$	$+$	0	$-$
direction of tangent			
shape of curve			

The stationary point $\left(1, \frac{1}{4}\right)$ is a maximum point.

Follow up 16

Solution

(a) $y = 8x + \dfrac{1}{2x^2}$

$= 8x + \dfrac{1}{2}x^{-2}$

Differentiate y with respect to x

$\dfrac{dy}{dx} = 8 - x^{-3}$

Differentiate $\dfrac{dy}{dx}$ with respect to x

$\dfrac{d^2y}{dx^2} = 3x^{-4}$

$= \dfrac{3}{x^4}$

(b) At stationary point, $\dfrac{dy}{dx} = 0.$

$8 - x^{-3} = 0$

$\dfrac{1}{x^3} = 8$

$x^3 = \dfrac{1}{8}$

$x = \dfrac{1}{2}$

Substituting $x = \dfrac{1}{2}$ into $y = 8x + \dfrac{1}{2x^2}$.

$y = 8(0.5) + \dfrac{1}{2(0.5)^2}$

$= 6$

The coordinates of the stationary point is $\left(\dfrac{1}{2}, 6\right)$.

Use second derivative to determine the nature of the stationary

Substituting $x = \dfrac{1}{2}$ into $\dfrac{d^2y}{dx^2} = 3x^{-4}$.

$\dfrac{d^2y}{dx^2} = 3\left(\dfrac{1}{2}\right)^{-4}$

$\dfrac{d^2y}{dx^2} = 48 > 0$

The stationary point $\left(\dfrac{1}{2}, 6\right)$ is a minimum point

Follow up 17

Solution

Let $\quad f(x) = x^3 - 1$ .... (1)

Differentitate (1) with respect to x

$\quad f'(x) = 3x^2$ .... (2)

At stationary point, $f'(x) = 0$.

$\therefore \quad 3x^2 = 0$

$\quad x = 0$

Substituting $x = 0$ into (1).

When $x = 0$, $f(0) = -1$.

$\therefore$ the stationary point is $(0, -1)$.

Differentitate (2) with respect to x

$\quad f''(x) = 6x$ .... (3)

When $x = 0$, $f''(0) = 0$

Differentitate (3) with respect to x

$\quad f'''(x) = 6$

Since nth derivative is odd, it implies that the stationary point is a point of inflection.

$\therefore (0, -1)$ is a point of inflextion.

Follow up 18

Solution

$$y = x + e^{1-2x}$$
$$\frac{dy}{dx} = 1 - 2e^{1-2x}$$

At stationary point, $\frac{dy}{dx} = 0$.

$$1 - 2e^{1-2x} = 0$$
$$2e^{1-2x} = 1$$
$$e^{1-2x} = \frac{1}{2} \quad \triangleleft \text{ take ln on both sides}$$
$$(1-2x)\ln e = \ln\left(\frac{1}{2}\right)$$
$$= \ln 1 - \ln 2$$
$$1 - 2x = -\ln 2$$
$$x = \frac{1+\ln 2}{2}$$

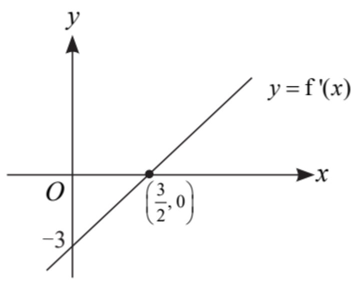
Substituting $x = \frac{1+\ln 2}{2}$ into $y = x + e^{1-2x}$.

$$y = \frac{1+\ln 2}{2} + e^{1-2\left(\frac{1+\ln 2}{2}\right)}$$
$$= \frac{1}{2} + \frac{1}{2}\ln 2 + e^{1-1-\ln 2}$$
$$= \frac{1}{2} + \frac{1}{2}\ln 2 + e^{-\ln 2}$$
$$= \frac{1}{2} + \frac{1}{2}\ln 2 + e^{\ln 2^{-1}}$$
$$= \frac{1}{2} + \frac{1}{2}\ln 2 + \frac{1}{2}$$
$$= 1 + \frac{\ln 2}{2}$$

The exact coordinates of the turning point of C are $\left(\frac{1+\ln 2}{2}, 1 + \frac{\ln 2}{2}\right)$.

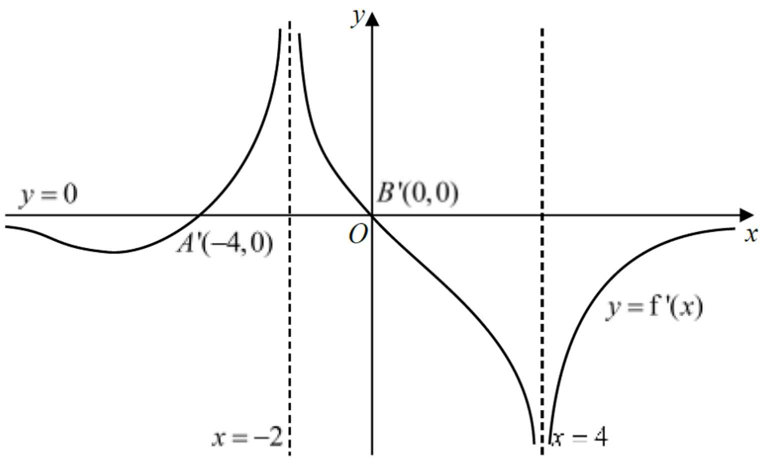
Follow up 19

Solution



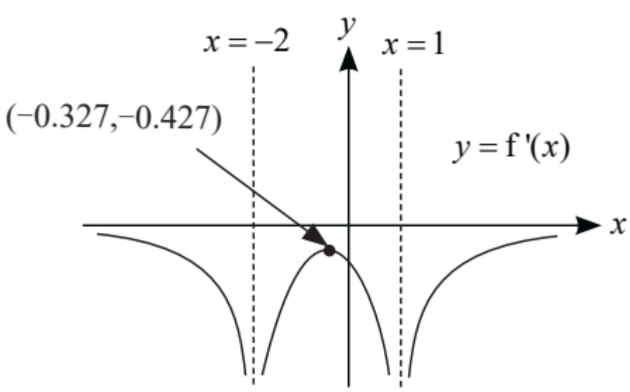
Follow up 20

Solution



Follow up 21

Solution

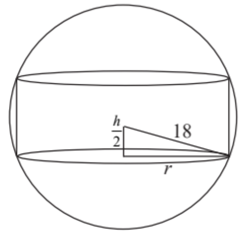


Follow up solutions C9 (1-4)

Follow up 1

Solution

Using Pythagoras Theorem



$$\left(\frac{h}{2}\right)^2 + r^2 = 18^2$$

$$\frac{h^2}{4} + r^2 = 324$$

$$h^2 + 4r^2 = 1296 \quad \text{(Shown) .... (1)}$$

(a) Differentiate (1) w.r.t. r

$$2h\frac{\mathrm{d}h}{\mathrm{d}r} + 8r = 0 \text{ .... (2)}$$

Given that the height of the cylinder, $h = 24$,
substituting $h = 24$ into (1) to find r.

$$(24)^2 + 4r^2 = 1296$$

$$r^2 = 180$$

$$r = 6\sqrt{5}$$

Substituting $h = 24$ and $r = 6\sqrt{5}$ into (2).

$$2(24)\frac{\mathrm{d}h}{\mathrm{d}r} + 8(6\sqrt{5}) = 0$$

$$2(24)\frac{\mathrm{d}h}{\mathrm{d}r} = -8(6\sqrt{5})$$

$$\frac{\mathrm{d}h}{\mathrm{d}r} = -\sqrt{5}$$

Given that the height is decreasing at a constant rate of $0.25\ \mathrm{cms}^{-1}$, i.e. $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{1}{4}$

Using Chain Rule,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$= \frac{1}{(-\sqrt{5})} \times \left(-\frac{1}{4}\right)$$

$$= \frac{1}{4\sqrt{5}}$$

$$= \frac{\sqrt{5}}{20}$$

The rate at which the radius is changing at this instant is $\frac{\sqrt{5}}{20}$ cm/s. (Shown)

(b) Let V be the volume of the cylinder.

$$V = \pi r^2 h \text{ .... (3)}$$

From (1): $r^2 = \dfrac{1296 - h^2}{4}$ .... (4)

Substituting (4) into (3).

$$V = \pi\left(\frac{1296 - h^2}{4}\right)h$$

$$V = \frac{\pi}{4}(1296h - h^3)$$

Differentiating V with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{4}(1296 - 3h^2)$$

When $h = 24$,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}h} &= \frac{\pi}{4}(1296 - 3h^2) \\ &= -108\pi\end{aligned}$$

Using Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= -108\pi \times \left(-\frac{1}{4}\right) \\ &= 27\pi\end{aligned}$$

The rate at which the volume of cylinder increasing when the height of cylinder is $24\,\text{cm}$ is $27\pi\ \text{cm}^3/\text{s}$.

Follow up 2

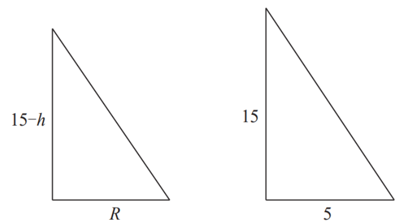
Solution

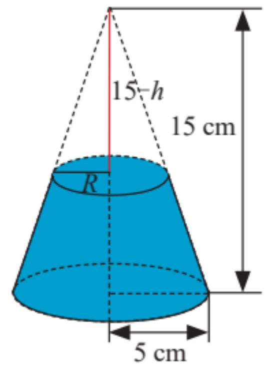
Let the depth of the water be h, the volume of water be V and the radius of the water level at that point be R.

Using similar triangle

$$\frac{15-h}{15}=\frac{R}{5}$$

$$R=\frac{15-h}{3}$$





$$V=\frac{1}{3}\pi(5)^2(15)-\frac{1}{3}\pi R^2(15-h)$$

$$=125\pi-\frac{1}{3}\pi\left(\frac{15-h}{3}\right)^2(15-h) \quad \triangleleft \text{ replace } R \text{ with } \frac{15-h}{3}$$

$$=125\pi-\frac{1}{27}\pi(15-h)^3$$

Differentiating V with respect to h.

$$\frac{dV}{dh}=-\frac{3}{27}\pi(15-h)^2(-1)$$

$$=\frac{\pi(15-h)^2}{9}$$

Given that water is leaking from the circular base of the cone at a rate of $10\text{ cm}^3\text{s}^{-1}$, i.e. $\frac{dV}{dt}=-10\text{cm}^3\text{s}^{-1}$.

Using chain rule,

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$

$$-10\text{cm}^3\text{s}^{-1}=\frac{\pi(15-h)^2}{9}\times\frac{dh}{dt}$$

At $h=12\text{cm}$,

$$-10\text{cm}^3\text{s}^{-1}=\frac{\pi(15-12)^2}{9}\times\frac{dh}{dt}$$

$$-10\text{cm}^3\text{s}^{-1}=\pi\times\frac{dh}{dt}$$

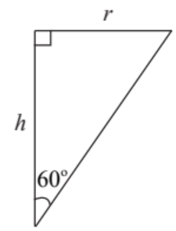
$$\frac{dh}{dt}=-\frac{10}{\pi}\text{cm}^3\text{s}^{-1}$$

The decreasing rate of the depth of water when the depth of water is 12 cm is $\frac{10}{\pi}\text{cm}^3\text{s}^{-1}$.

Follow up 3

Solution

Volume of water in the conical tank at time t seconds be V.



$$V = \frac{1}{3}\pi r^2 h \text{ .... (1)}$$

Refer to the diagram, using trigonometric ratio

$$\tan 60^\circ = \frac{r}{h}$$

$$r = h\sqrt{3} \text{ .... (2)}$$

Substituting (2) into (1).

Therefore $\quad V = \pi h^3$ .... (3)

Differentiating V respect to t.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 3\pi h^2 \frac{\mathrm{d}h}{\mathrm{d}t} \text{ .... (4)}$$

Volume of water in the conical tank = Volume of water filled − Volume of water flowing out

$$V = 94\pi - (2\pi \text{ per second})(t)$$

When $t = 15$,

$$V = 94\pi - (2\pi)(15)$$
$$= 64\pi$$

Substituting $V = 64\pi$ into (3).

$$\pi h^3 = 64\pi$$
$$h = 4$$

Given that water begins to flow out at a constant rate of 2π cm^{3}s^{-1}, $\frac{\mathrm{d}V}{\mathrm{d}t} = -2\pi$.

Substituting $h = 4$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = -2\pi$ into (4).

$$-2\pi = 3\pi(4)^2 \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\therefore \quad \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-2\pi}{3\pi(4)^2}$$

$$= -\frac{1}{24}$$

$\therefore$ Rate at which h is decreasing at the instant when $t = 15$ is $\frac{1}{24}$ cms^{-1}.

Learning point:

Some common errors from above solution:

1. $h^3 = 64 \Rightarrow h = 8$
2. $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-2\pi}{3\pi h^2} = \frac{-2}{3}h^2$
3. $\frac{\mathrm{d}V}{\mathrm{d}t} = 2\pi$
4. Differentiating V without realising that both h and r are variables:

   Eg : $V = \frac{1}{3}\pi r^2 h$

   $\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi r^2$ or $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2}{3}\pi r h$
5. Wrote either "$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{24}$" or "Rate of decrease of $h = -\frac{1}{24}$".

Follow up 4

Solution

Given $x = \cos p$ $\quad\quad$ $y = \sin^3 p$

$$\frac{\mathrm{d}y}{\mathrm{d}p} = 3\cos p \sin^2 p \qquad \frac{\mathrm{d}x}{\mathrm{d}p} = -\sin p$$

Using chain rule,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}p} \times \frac{\mathrm{d}p}{\mathrm{d}x} \\
&= \frac{3\cos p \sin^2 p}{-\sin p} \\
&= -3\sin p \cos p \\
&= \frac{-3(2\sin p \cos p)}{2} \qquad \triangleleft \text{ use double angle, } \sin 2p = 2\sin p \cos p \\
&= -\frac{3}{2}\sin 2p
\end{aligned}$$

Let $z = \frac{\mathrm{d}y}{\mathrm{d}x}$

$\therefore$
$$z = -\frac{3}{2}\sin 2p$$

$$\frac{\mathrm{d}z}{\mathrm{d}p} = -3\cos 2p$$

Using chain rule,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}p} \times \frac{\mathrm{d}p}{\mathrm{d}t}$$

Given that p is increasing at rate of 0.5 units per second, i.e. $\frac{\mathrm{d}p}{\mathrm{d}t} = 0.5$.

$$\begin{aligned}
&= -3\cos 2p \times (0.5) \\
&= -\frac{3}{2}\cos 2p
\end{aligned}$$

When $p = \frac{\pi}{3}$

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= -\frac{3}{2}\cos\left(\frac{2\pi}{3}\right) \\
&= 0.75
\end{aligned}$$

Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing at 0.75 units per second when $p = \frac{\pi}{3}$.

Follow up solutions C10 (1-4)

Follow up 1

Solution

(a) $V = \frac{1}{3}\pi(PC)^2(AP)$ .... (1) ⊲ formulae of cone, $V = \frac{1}{3}\pi r^2$

In ΔAOR, $\sin\theta = \frac{r}{AO}$

$AO = \frac{r}{\sin\theta}$

$AO = r\,\mathrm{cosec}\,\theta$ .... (2)

$\therefore AP = r + r\,\mathrm{cosec}\,\theta$

In ΔAPC, $\tan\theta = \frac{PC}{AP}$

$PC = AP(\tan\theta)$

$= (OP + AO)(\tan\theta)$ ⊲ OP = radius of the sphere = r

$= (r + r\,\mathrm{cosec}\,\theta)(\tan\theta)$

$= r(1+\mathrm{cosec}\,\theta)\tan\theta$ .... (3)

$V = \frac{1}{3}\pi[r(1+\mathrm{cosec}\,\theta)\tan\theta]^2\ r(1+\mathrm{cosec}\,\theta)$

$V = \frac{1}{3}\pi r^3(1+\mathrm{cosec}\,\theta)^3\tan^2\theta$ (Shown) .... (4)

(b) Differentiate (4) with respect to θ

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{1}{3}\pi r^3[3(1+\mathrm{cosec}\,\theta)^2(-\mathrm{cosec}\,\theta\cot\theta)\tan^2\theta + (1+\mathrm{cosec}\,\theta)^3 2\tan\theta\sec^2\theta]$ ⊲ use product rule

$= \frac{1}{3}\pi r^3(1+\mathrm{cosec}\,\theta)^2[-3\mathrm{cosec}\,\theta\tan\theta + (1+\mathrm{cosec}\,\theta)2\tan\theta\sec^2\theta]$ ⊲ factor out $(1+\mathrm{cosec}\,\theta)^2$

At stationary, $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$.

$\frac{1}{3}\pi r^3(1+\mathrm{cosec}\,\theta)^2[-3\mathrm{cosec}\,\theta\tan\theta + (1+\mathrm{cosec}\,\theta)2\tan\theta\sec^2\theta] = 0$

$\frac{1}{3}\pi r^3(1+\mathrm{cosec}\,\theta)^2 = 0$ or $-3\left(\frac{1}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) + \left(1+\frac{1}{\sin\theta}\right)\left(\frac{2\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos^2\theta}\right) = 0$

$\mathrm{cosec}\,\theta = -1$ or $-\frac{3}{\cos\theta} + \left(\frac{\sin\theta+1}{\sin\theta}\right)\left(\frac{2\sin\theta}{\cos^3\theta}\right) = 0$

$\sin\theta = -1$ or $-\frac{3}{\cos\theta} + \frac{2\sin\theta}{\cos^3\theta} + \frac{2}{\cos^3\theta} = 0$

(rejected since θ is acute)

Consider $-\dfrac{3}{\cos\theta}+\dfrac{2\sin\theta}{\cos^3\theta}+\dfrac{2}{\cos^3\theta}=0$

$$-3\cos^2\theta+2\sin\theta+2=0$$

$$-3(1-\sin^2\theta)+2\sin\theta+2=0$$

$$3\sin^2\theta+2\sin\theta-1=0$$

$$(\sin\theta+1)(3\sin\theta-1)=0$$

$\sin\theta=-1$ (rejected since θ is acute) or $\sin\theta=\dfrac{1}{3}$ (Shown)

When $\sin\theta=\dfrac{1}{3}$, from GC,

$$\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}=75.4r^3>0 \text{ since } r>0$$

$\therefore$ V is minimum when $\sin\theta=\dfrac{1}{3}$.

Alternative Method

θ	$\left(\sin^{-1}\frac{1}{3}\right)^-$ $=0.339$	$\sin^{-1}\frac{1}{3}$ $=0.340$	$\left(\sin^{-1}\frac{1}{3}\right)^+$ $=0.341$
$\frac{\mathrm{d}V}{\mathrm{d}\theta}$	$-0.0633r^3$ <0	0	$0.0874r^3$ >0

$\therefore$ V is minimum when $\sin\theta=\dfrac{1}{3}$.

Follow up 2

Solution

(a) $BD = \sqrt{n^2 + n^2}$ ◁ using Pythagoras Theorem

$= \sqrt{2}n$

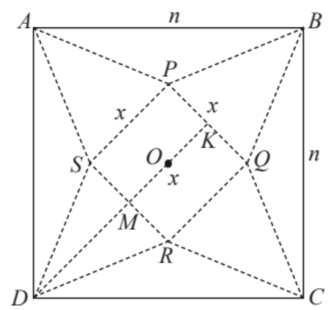


Figure 1

Let M be the midpoint of SR and K be the midpoint of PQ.

$\therefore \quad DM = \dfrac{BD - BK}{2}$

$= \dfrac{\sqrt{2}n - x}{2}$

Note that $DM = EM$.

$EM^2 = OE^2 + OM^2$ ◁ using Pythagoras Theorem

$OE^2 = EM^2 - OM^2$

$= \left(\dfrac{\sqrt{2}n - x}{2}\right)^2 - \left(\dfrac{x}{2}\right)^2$

$= \dfrac{(\sqrt{2}n - x)^2}{4} - \dfrac{x^2}{4}$

$= \dfrac{1}{4}(2n^2 - 2\sqrt{2}nx + x^2 - x^2)$

$= \dfrac{1}{4}(2n^2 - 2\sqrt{2}nx)$

$= \dfrac{1}{2}(n^2 - \sqrt{2}nx)$ (Shown)

(b) $V = \dfrac{1}{3}$(base area)(height, OE)

$= \dfrac{1}{3}x^2\sqrt{\dfrac{1}{2}(n^2 - \sqrt{2}nx)}$

$V^2 = \dfrac{x^4}{18}(n^2 - \sqrt{2}nx)$ (shown) .... (1)

$$V^2 = \frac{x^4}{18}(n^2 - \sqrt{2}nx)$$

$$= \frac{n^2}{18}x^4 - \frac{\sqrt{2}n}{18}x^5$$

Differentiate (1) with respect to *x*.

$$2V\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{n^2}{18}(4x^3) - \frac{\sqrt{2}n}{18}(5x^4) \text{ .... (2)}$$

$$= \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4$$

At stationary $\frac{\mathrm{d}V}{\mathrm{d}x} = 0.$

$$\frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4 = 0$$

$$\frac{nx^3}{18}(4n - 5\sqrt{2}x) = 0$$

$$x = \frac{4n}{5\sqrt{2}} \quad \text{or} \quad x = 0 \text{ (rejected, } \because \ x > 0)$$

$$= \frac{2\sqrt{2}}{5}n$$

To determine the nature of the stationary point

Method 1 (2nd derivative Test)

From (2): $2V\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4$

Differentiate (2) with respect to *x*.

$$2\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{9}(3x^2) - \frac{5\sqrt{2}n}{18}(4x^3)$$

$$2\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}x^2 - \frac{10\sqrt{2}n}{9}x^3 \text{ .... (2)}$$

Substituting $x = \frac{2\sqrt{2}n}{5}$ and $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ into (2)

$$\therefore \quad 2(0)^2 + 2V\frac{d^2V}{dx^2} = \frac{2n^2}{3}\left(\frac{2\sqrt{2}n}{5}\right)^2 - \frac{10\sqrt{2}n}{9}\left(\frac{2\sqrt{2}n}{5}\right)^3$$

$$2V\frac{d^2V}{dx^2} = \frac{2n^2}{3}\left(\frac{8n^2}{25}\right) - \frac{10\sqrt{2}n}{9}\left(\frac{16\sqrt{2}n^3}{125}\right)$$

$$= \frac{16n^4}{75} - \frac{64x^4}{225}$$

$$= -\frac{16n^4}{225}$$

$$\frac{d^2V}{dx^2} = -\frac{8n^4}{225V} < 0 \text{ (since } V, n > 0)$$

$\therefore$ V is maximum when $x = \dfrac{2\sqrt{2}}{5}n$.

Method 2 (1st derivative Test)

x	$\left(\frac{2\sqrt{2}}{5}n\right)^-$	$\frac{2\sqrt{2}}{5}n$	$\left(\frac{2\sqrt{2}}{5}n\right)^+$
Explanation	$4n - 5\sqrt{2}x > 0$ $x > 0$	$4n - 5\sqrt{2}x = 0$	$4n - 5\sqrt{2}x < 0$ $x > 0$
$\frac{dV}{dx}$	+ve	0	−ve
Slope	/	–	\

$\therefore$ V is maximum when $x = \dfrac{2\sqrt{2}}{5}n$

(c) Method I (Analytical)

Given that the volume of the pyramid is greater than 45 cm^3

i.e. $V > 45$

From part **(b)** $V^2 = \dfrac{x^4}{18}(n^2 - \sqrt{2}nx)$

$\therefore \quad \dfrac{x^4}{18}(n^2 - \sqrt{2}nx) > 45^2$

V is maximum when $x = \dfrac{2\sqrt{2}}{5}n$.

$$\frac{n^2}{18}\left(\frac{2\sqrt{2}}{5}n\right)^4 - \frac{\sqrt{2}n}{18}\left(\frac{2\sqrt{2}}{5}n\right)^5 > 45^2$$

$$\frac{32}{5625}n^6 - \frac{128}{28125}n^6 > 2025$$

$$\frac{32}{28125}n^6 > 2025$$

$n > 11.008$ (5s.f.)

The least integer n is 12. The smallest cardboard given is $12\text{cm} \times 12\text{cm}$.

Method II (GC table method)

From part **(b)**, $V^2 = \dfrac{x^4}{18}\left(n^2 - \sqrt{2}nx\right)$

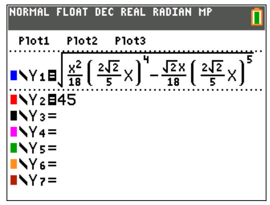
$$V = \sqrt{\frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right)}$$

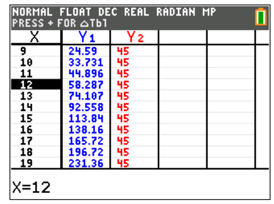
Given that the volume of the pyramid is greater than 45 cm^3

i.e. $V > 45$.

V is maximum when $x = \dfrac{2\sqrt{2}}{5}n$

$$\sqrt{\frac{n^2}{18}\left(\frac{2\sqrt{2}}{5}n\right)^4 - \frac{\sqrt{2}n}{18}\left(\frac{2\sqrt{2}}{5}n\right)^5} > 45$$





Using GC,

When $n = 11, V = 44.9 < 45$

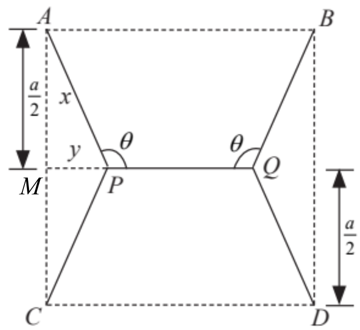
When $n = 12, V = 58.3 > 45$

Smallest cardboard given is $12\text{cm} \times 12\text{cm}$.

Follow up 3

Solution

(a) Let x be AP, y be RP and M be mid-point of AC.
In ΔAMP



$$\sin(\pi-\theta)=\frac{\frac{a}{2}}{x}$$

$$x=\frac{a}{2\sin\theta}$$

$$\tan(\pi-\theta)=\frac{\frac{a}{2}}{y}$$

$$y=\frac{a}{2\tan\theta}$$

$$T=(AP+BP+DQ+CP)+PQ$$

$$=4\left(\frac{a}{2\sin\theta}\right)+\left[a-2\left(-\frac{a}{2\tan\theta}\right)\right]$$

$$=2a\operatorname{cosec}\theta+a+a\cot\theta$$

$$=a(2\operatorname{cosec}\theta+1+\cot\theta)$$

(b) $\frac{\mathrm{d}T}{\mathrm{d}\theta}=-2a\operatorname{cosec}\theta\cot\theta-a\operatorname{cosec}^2\theta$

At stationary, $\frac{\mathrm{d}T}{\mathrm{d}\theta}=0$

$$-2a\operatorname{cosec}\theta\cot\theta-a\operatorname{cosec}^2\theta=0$$

$$\operatorname{cosec}\theta(2\cot\theta+\operatorname{cosec}\theta)=0$$

$$\cos\theta=-\frac{1}{2}\quad\text{or}\quad\operatorname{cosec}\theta=0\ \text{(undefined)}$$

$$\theta=\frac{2\pi}{3}$$

Using First Derivative Test

θ	$\left(\frac{2\pi}{3}\right)^-$	$\frac{2\pi}{3}$	$\left(\frac{2\pi}{3}\right)^+$
$\frac{\mathrm{d}T}{\mathrm{d}\theta}$	Negative	0	Positive

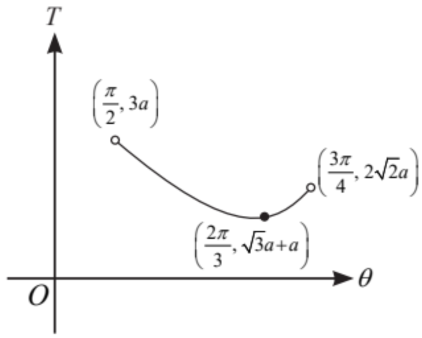
Hence, T is minimum when $\theta=\frac{2\pi}{3}$

$$T_{\min}=2a\operatorname{cosec}\frac{2\pi}{3}+a+a\cot\frac{2\pi}{3}$$

$$=\sqrt{3}a+a$$

$\therefore$ minimum value of T is $(\sqrt{3}+1)a$

(c) Graph of T against θ



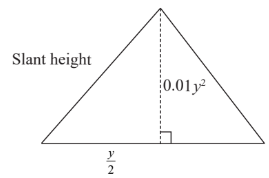
Follow up 4

Solution

(a) Floor area, $xy = 4000$

$$y = \frac{4000}{x}$$

Slant height of the rectangular roof $= \sqrt{(0.01y^2)^2 + \left(\frac{y}{2}\right)^2}$



Area $A = 2(\text{vertical sides}) + 2(\text{rectangular roofs})$

$$= 2(4x) + 2x\sqrt{(0.01y^2)^2 + \left(\frac{y}{2}\right)^2} \quad \triangleleft \text{ replace } y = \frac{4000}{x}$$

$$= 8x + 2x\sqrt{\left(0.01 \times \frac{4000^2}{x^2}\right)^2 + \left(\frac{4000}{2x}\right)^2}$$

$$= 8x + 2x\sqrt{\frac{160000^2}{x^4} + \frac{2000^2}{x^2}}$$

$$= 8x + 2x \times \sqrt{\frac{2000^2}{x^2}}\sqrt{\frac{6400^2}{x^2} + \frac{1}{1}} \quad \triangleleft \text{ factor out } y = \sqrt{\frac{2000^2}{x^2}}$$

$$= 8x + 4000\sqrt{\frac{6400}{x^2} + 1} \quad \text{(Shown)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(b) Differentiate (1) with respect to x.

$$\frac{dA}{dx} = 8 + 2000\left(\frac{6400}{x^2} + 1\right)^{-\frac{1}{2}}\left(-\frac{12800}{x^3}\right)$$

At stationary, $\frac{dA}{dx} = 0$.

i.e. $8 + 2000\left(\frac{6400}{x^2} + 1\right)^{-\frac{1}{2}}\left(-\frac{12800}{x^3}\right) = 0$

$$\left(\frac{6400}{x^2} + 1\right)^{-\frac{1}{2}}\frac{25600000}{x^3} = 8$$

$$\frac{25600000}{8x^3} = \left(\frac{6400}{x^2} + 1\right)^{\frac{1}{2}}$$

$$\left(\frac{3200000}{x^3}\right)^2 = \frac{6400}{x^2} + 1$$

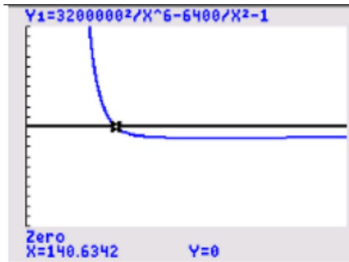
$$\frac{1.024 \times 10^{13}}{x^6} = \frac{6400}{x^2} + 1 \quad \text{(Shown)}$$

$\therefore\ a = 1.024 \times 10^{13}$

Use GC to solve $\dfrac{1.024\times 10^{13}}{x^6} = \dfrac{6400}{x^2} + 1$.

$\therefore\ x = 140.6342$

$\quad = 141$ (to 3 sf)



Method 1 (Using First Derivative Test)

x	140.63^-	140.63	140.63^+
$\dfrac{dA}{dx}$	$-$ ve	0	+ ve

$\therefore$ when $x = 140.6342$, $V\ A$ is minimum.

Method 2 (Using Second Derivative Test)

$$\frac{d^2A}{dx^2} = \left(\frac{6400}{x^2}+1\right)^{-\frac{1}{2}}\left(\frac{73800000}{x^4}\right)$$

$$= \left(\frac{6400}{x^2}+1\right)^{-\frac{3}{2}}\left(-\frac{12800000}{x^3}\right)\left(-\frac{12800}{x^3}\right) > 0$$

When $x = 140.6342$

$$= \left(\frac{6400}{140.6342^2}+1\right)^{-\frac{3}{2}}\left(-\frac{12800000}{140.6342^3}\right)\left(-\frac{12800}{140.6342^3}\right) > 0$$

Hence A is minimum when $x = 140.6342$

Minimum Cost

$$= \left[8(140.6342) + 4000\sqrt{\frac{6400}{140.6342^2}+1}\right]2.50$$

$= \$14317.43$ (nearest cents)

$= \$14300$ (to 3 s.f.)

The minimum total cost of the material for the whole tent is $14300.